Last-Call Auctions with Asymmetric Bidders

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Abstract

Favoring a bidder through a Right of First Refusal (ROFR) in First-Price Auctions is not only common practice in industrial procurement auctions, but can also be a meaningful tool to increase the auctioneer's expected revenue. We compare Last-Call Auctions (i.e. First-price Auctions with ROFR) to standard Second-price Auctions with asymmetric bidders, where the bidders' strengths are modeled by either linear, strictly convex or strictly concave beta distributions. We show that if the asymmetry between bidders is sufficiently high and the weak bidder is weak enough, the Last-Call Auction can outperform a standard Second-Price Auction in terms of expected auction revenue.

Our analysis is based on the work of Arozamena and Weinschelbaum (2009) and yields that for specific value distributions of the favored bidder, the non-favored bidder's optimal bid is more aggressive in a Last-Call Auction than in a First-Price Auction without favoritism. We show that a profit-maximizing auctioneer always (weakly) prefers to favor to favor the weak bidder. However, for most combinations of asymmetric bidders the expected auction revenue remains the same independent whether the weak or strong bidder is favored. Furthermore, we characterize combinations of bidders' value distributions in which the auctioneer gains a higher expected profit by granting a ROFR to the weak bidder in a First-Price Auction instead of conducting a Second-Price Auction without favoritism.

Keywords: auctions, industrial procurement, asymmetric bidders, right of first refusal, favoritism

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1. Introduction

In auctions different forms of favoritism can be established in order to accommodate the individual relationship between seller and buyer. We focus on favoritism through the assignment of a so called Right of First Refusal (ROFR). This kind of favoritism is especially used for long-term business partners and grants an exceptional position in the selling or procurement process. For example, if a firm plans to procure certain products via a procurement auction, but wants to protect their favorite long-term supplier from the competition of potentially unknown market-entrants she can assign a ROFR to her favorite supplier. That is, the favored supplier must not participate in the competitive bidding process, but has the chance to match the winning bid afterwards. A broad variety of practices of ROFR can be found in Walker (1999).

The scientific literature examines in First-Price Auctions as well as in Second-Price Auctions several impacts of granting a Right of First Refusal on bidding behavior and initial auction goals as expected auction revenue and efficiency. Bikhchandani et al. (2005) state that this form of favoritism will never be advantageous in terms of increased auction revenue and even may lead to inefficient outcomes in Second-Price Auctions. All mentioned authors in the following examine First-Price Auctions with ROFR. Though most of them consider the coalition of auctioneer and favored bidder and hence only investigate the joint surplus of both. For example, Choi (2009) states for two symmetric bidders that the joint surplus of auctioneer and favored bidder can be increased by the assignement of a ROFR, however, only at the expense of the third party's payoff. Burguet and Perry (2007) find that the auctioneer may benefit in a procurement auction with two asymmetric bidders from granting a ROFR combined with certain forms of bribery. In contrast to those, we aim to find situations in which the auctioneer's expected revenue increases independent from potential compensation payments by the favored bidder or regarding a joint surplus of favored bidder and auctioneer. Consequently, in our work the auctioneer's revenue is analyzed in isolation. This approach is also adopted by Brisset et al. (2012), who show that heterogeneous risk attitudes of the bidders may be the crucial factor for an increased auction revenue. Furthermore, Lee (2008) demonstrates that a certain degree of asymmetry among bidders' strengths yields a higher expected profit for the auctioneer in a First-Price Auction with assigning a ROFR than without. Related to his work we address the question under which assumptions regarding two asymmetric bidders the auctioneer can benefit from favoring one of the bidders in First-Price Auctions compared to incentive compatible Second-Price Auctions. For that, we assume different forms of asymmetries between the participating bidders. Beyond the work of Lee (2008), who defines the asymmetry by uniform distributions on staggered intervals, we model the bidders' value distributions on a common interval by linear, strictly convex and strictly concave beta distributions. According to the work of Arozamena and Weinschelbaum (2009) the curvature of the favored bidder's value distribution may play a decisive role with respect to the aggressiveness of the non-favored bidder's bidding behavior. Furthermore, we find an increase in the expected auction revenue in case of asymmetric bidders - depending on the non-favored bidder's value distribution . [1]

2. Model

We define a Last-Call Auction as a First-Price sales Auction, in which the auctioneer favors one of the bidders by awarding a *Right of First Refusal*.[2] In a sales auction, the Right of First Refusal, hereinafter referred to as ROFR, offers the favored bidder the option to buy the good at the best price submitted by the competing bidders. After the auctioneer has chosen a favored bidder and proclaimed her decision to all participants, a sealed-bid First-Price Auction is conducted. Hence the highest submitted bid determines the price the winner has to pay. However, the highest bidder will only win the auction only if the favored bidder does not exercise the ROFR. In case the favored bidder exercises her ROFR and so accepts the highest bid, she will win the auction and acquire the good at the resulting price. Thus, the resulting price is always the highest submitted bid in a Last-Call Auction, the winner, however, can either be the favored bidder or the highest non-favored bidder, if the favored bidder declines to exercise the ROFR. Instead of to exercise the ROFR we also say *to match* the winning bid. Further, it is to emphasize that the favored bidder only is allowed to match, if her initial bid was lower than the winning bid or she even did not submit any initial bid at

[1] We act on the assumption of a two-bidder case, i.e. one favored and one non-favored bidder.
[2] Our results can easily be modified for procurement auctions.

all.[3] We limit our work on the following mechanism: The favored bidder does not submit any initial bid, but only decides at the second stage whether to accept the winning bid or not. This basic auction mechanism is deduced from the work of Lee (2008).

As already discussed by Güth and Van Damme (1986) a Last-Call Auction with two bidders can be interpreted as an auction, where the situation of the non-favored bidder corresponds to that in a First-Price Auction and the favored bidder's situation to that in a Second-Price Auction. The non-favored bidder determines the price she has to pay in case of winning through her submitted bid and the favored bidder decides whether to match her opponent's bid. [4]

Our analysis focuses on a two-bidder case for Last-Call Auctions. Accordingly one non-favored bidder I and one favored bidder II compete against each other. We suppose an independent private value model, i.e. both bidders assign values x_I and x_{II} to the good, which are private information and independent of each other. We restrict our analysis to risk-neutral bidders and cases in which distributions of both bidders F_I and F_{II} are either linear, strictly concave or strictly convex beta distributions with support on $[0, 1]$ and publicly known. We assume that the auctioneer does not assign a positive value to the good, i.e. $x_0 = 0$..The number of bidders (here $n = 2$) as well as the fact that bidders are risk-neutral is common knowledge. In the course of the work this model is preserved as far as either symmetric or asymmetric bidders are supposed. We call a bidder stronger than her opponent if her value distribution dominates her opponents' one acccording to the reverse hazard rate order.

3. Analysis

The non-favored bidder's bid b_I always determines the price in the two bidder case since we suppose that the favored bidder II does not submit an initial bid, but only matches the non-favored bidder's bid if applicable. Thus, first the equilibrium bidding strategy of the non-favored and hence price-determining bidder is calculated. Afterwards we will demonstrate how the information about the favored bidder's strength affects the non-favored bidder's bid.

[3] A Last-Call Auction can be considered as a two-stage mechanism, where at the first stage a First-Price Auction is conducted and at the second stage the favored bidder has the option to match the winning bid.

[4] In particular, supposing two symmetric bidders, whose valuations are uniformly distributed on $[0, 1]$, the equilibrium bidding function of the non-favored bidder in a Last-Call Auction is exactly the same as in a First-Price Auction. Further, the situation of the favored bidder corresponds exactly to that in a Second-Price Auction as well.

Proposition 1. *The non-favored bidder I's equilibrium bidding strategy $\beta_I : x_I \mapsto b_I$ in a Last-Call Auction is*

$$\beta_I(x_I) = x_I - \frac{F_{II}(\beta_I(x_I))}{f_{II}(\beta_I(x_I))}, \tag{1}$$

where x_I is bidder I's valuation and the favored bidder's value distribution and density functions are given by F_{II} and f_{II}.

Proof. The expected profit of bidder I is the difference between her valuation x_I and her bid $b_I = \beta_I(x_I)$ in case of winning, that is, if b_I exceeds the favored bidder's valuation x_{II} and consequently she declines to match. If $b_I < x_{II}$, the favored bidder will match and consequently bidder I's profit is zero. Let F_{II} be the distribution function of the favored bidder's valuation. It follows

$$E[\pi_I] = (x_I - \beta_I(x_I))P(X_{II} \leq \beta_I(x_I)) = (x_I - \beta_I(x_I))F_{II}(\beta_I(x_I)).$$

Suppose that bidder I wants to maximize her expected profit through her submitted bid $\beta_I(x_I)$. With the first-order condition follows

$$\frac{\partial}{\partial \beta_I(x_I)} E[\pi_I] = x_I f_{II}(\beta_I(x_I)) - F_{II}(\beta_I(x_I)) - \beta_I(x_I) f_{II}(\beta_I(x_I)) \overset{!}{=} 0,$$
$$\Leftrightarrow \beta_I(x_I) = x_I - \frac{F_{II}(\beta_I(x_I))}{f_{II}(\beta_I(x_I))}.$$

□

Consequently, the non-favored bidder always shades her bid in equilibrium. By maximizing the expected rent the non-favored bidder finds herself in a trade-off situation: On the one hand a higher bid increases her winning probability. On the other hand, a higher bid reduces her profit in case of winning, because she determines the payment through this bid. So the equilibrium bidding strategy balances these opposite effects to maximize the bidder's expected rent. The non-favored bidder's equilibrium bidding behavior is in the further analysis easier to handle by utilizing the explicit inverse equilibrium bidding function $\beta_I^{-1} : b_I \mapsto x_I$ instead of the implicit equilibrium bidding function presented above. Therefore we will demonstrate below that the equilibrium bidding function is strictly monotone and therefore bijective and invertible for strictly concave, convex and linear beta value distribution functions.

Remark 1. *If $\beta_I(x_I)$ is bijective, the inverse equilibrium bidding strategy of the non-favored and price-determining bidder I in a Last-Call Auction is given by*

$$\beta_I^{-1}(p) = p + \frac{F_{II}(p)}{f_{II}(p)},$$

where $p \in [0,1]$ is the resulting price and F_{II} the favored bidder II's value distribution with corresponding density f_{II}.

Proof. This inverse equilibrium bidding strategy follows immediately from equation 1, where p equals the non-favored bidder's bid $\beta_I(x_I)$ and further $x_1 = \beta_I^{-1}(p)$. □

Remark 2. *Let the favored bidder's value distribution F_{II} be a linear, strictly concave or strictly convex beta distribution. Then the equilibrium bidding strategy of the non-favored bidder in a Last-Call Auction $\beta_I(x_I)$ is strictly monotone and hence bijective.*

Proof. The monotony of $\beta_I(x_I)$ is implied by the monotony of $\beta_I^{-1}(p)$, i.e. by $\frac{\partial}{\partial p}\beta_I^{-1}(p) > 0$ $\forall p \in [0,1]$. First a linear beta distribution $F_{II}(p) = p$ is supposed for the favored bidder's value distribution. Then differentiating $\beta_I^{-1}(p) = 2p$ with respect to p ensues

$$\frac{\partial}{\partial p}\beta_I^{-1}(p) = 2 > 0 \ \ \forall p \in [0,1].$$

Assuming F_{II} is a strictly convex beta distribution for the derivative of $\beta_I^{-1}(p) = \frac{\alpha}{\alpha+1}p$ with respect to p follows for $\alpha > 1$

$$\frac{\partial}{\partial p}\beta_I^{-1}(p) = \frac{\alpha}{\alpha+1} > 0 \ \ \forall p \in [0,1].$$

Finally, if F_{II} is a strictly concave beta distribution after differentiating $\beta_I^{-1}(p) = p + \frac{1-(1-p)^\gamma}{\gamma(1-p)^{\gamma-1}}$ it holds for $\gamma > 1$

$$\frac{\partial}{\partial p}\beta_I^{-1}(p) = 1 + \frac{1}{\gamma}\left(1 - \frac{1-\gamma}{(1-p)^\gamma}\right) > 0 \ \ \forall p \in [0,1].$$

□

The inverse equilibrium bidding strategy depends on the favored bidder's value distribution F_{II} and density function f_{II}, which are common knowledge. That is the price determining bid $\beta_I(x_I)$ is influenced by the strength of the competing favored bidder. The stronger the favored bidder the more aggressive is the non-favored bidder's submitted bid, i.e. a stronger opponent will lead the non-favored bidder to offer a higher price.

The fact that the non-favored bidder offers a higher price if the strength of her opponent increases is intuitive: a stronger opponent will lower the winning probability and the non-favored bidder attends to compensate this effect by bidding more aggressively.

In the next step, the expected auction revenue in a Last-Call Auction is deduced. On the one hand the non-favored bidder's equilibrium bid depends on her individual valuation and on the other hand it is influenced by the strength of the competing favored bidder. Consequently, the expected auction revenue, finally determined by the non-favored bidder's bid, is affected by both bidders' strengths.

Proposition 2. *The distribution function of the expected payment in a Last-Call-Auction is given by*

$$F^{LCA}(p) = F_I(\beta_I^{-1}(p)),$$

where $\beta_I^{-1}(p) = p + \frac{F_{II}(p)}{f_{II}(p)}$ *is the inverse equilibrium bidding strategy of the non-favored bidder* I *and* $p \in [0,1]$.

Proof. The distribution function $F^{LCA}(p)$ is the probability that the expected auction revenue is lower than or equal to p. That is, the probability that the price-determining bid $b_I = \beta_I(x_I)$, where x_I is the non-favored bidder's valuation, is lower than or equal to p. Therefore the distribution function $F^{LCA}(p)$ corresponds to the probability $P(b_I \leq p) = P(X_I \leq \beta_I^{-1}(p)) = F_I(\beta_I^{-1}(p))$. □

Proposition 3. *The expected auction revenue in a Last-Call Auction is*

$$E[p^{LCA}] = \int_0^\infty 1 - F^{LCA}(p)dp = \int_0^\infty 1 - F_I(\beta_I^{-1}(p))dp \tag{2}$$

Proof. Let the distribution of the expected payment in the Last-Call Auction be given by $F^{LCA}(p)$. Then for any $p \in [0,1]$ Proposition (2) yields the assertion. □

Notice that proposition 3 only applies if the favored bidder was selective elected and not if one of the bidders is favored by chance.

In the following, we demonstrate under particular assumptions that for asymmetries between bidders' strengths hat the bidding behavior of the non-favored bidder changes and as a consequence the expected auction revenue in a Last-Call Auction may exceed that in a Second-Price Auction. Consequently, the kind of asymmetry between bidders is a crucial factor for the auctioneer to decide whether to conduct a Last-Call or a Second-Price Auction. [5] For that purpose, two asymmetric bidders are considered, one strong bidder and one weak bidder. The bidders' valuations will be drawn independently from the same interval $[0,1]$, where the weak bidder's distribution F_w on $[0,1]$ is stochastically dominated by the strong bidder's distribution F_s on $[0,1]$ according to the reverse hazard-rate order. Further it holds that F_s first-order stochastically dominates F_w, i.e. $F_s(x) \leq F_w(x)$ for all $x \in [0,1]$, and therefore $E[X_w] \leq E[X_s]$. That is, the expected valuation of the strong bidder is higher than the weak bidder's expected valuation for the good. Further it is assumed that the value distributions are either linear, strictly convex or strictly concave beta distributions.

3.1. Impact of the ROFR on bidding behavior

According to Bagnoli and Bergstrom (2005) linear, strictly concave or strictly convex beta distributions are logconcave. Arozamena and Weinschelbaum (2009) find that for logconcave value distributions symmetric bidders may bid more or less aggressive in a Last-Call Auction than in a First-Price Auction depending on the ratio $\rho(x) = \frac{F(x)}{f(x)}$: If $\rho(x)$ is strictly concave (convex) in x, symmetric bidders bid more (less) aggressively, whereas the bidding behavior remains unaltered in case $\rho(x)$ is linear in x.

Remark 3. *The bidding behavior in a Last-Call Auction corresponds to that in a First-Price Auction in case the non-favored (price determining) bidder faces an opponent with a linear or strictly convex beta distribution. In case the non-favored bidder's opponent has a strictly concave beta distribution the bidding behavior is more aggressive in a Last-Call Auction than in a First-Price Auction.*

[5] In the symmetric case, the auctioneer does not benefit from granting a ROFR to any bidder for the considered combinations of beta distributions.

Proof. As we consider an asymmetric two-bidder case, the strength of the favored bidder is crucial for the bidding strategy of the non-favored and price determining bidder. Hence, the favored bidder's value distribution and density function are relevant for the ratio $\rho(x) = \frac{F(x)}{f(x)}$. Let $F(x) = x$ and $\tilde{F}(x) = x^{\alpha}$ be the favored bidders' value distributions, where $\alpha > 1$. Then both value distributions are logconcave, see Bagnoli and Bergstrom (2005), and it follows

$$\rho(x) = \frac{F(x)}{f(x)} = \frac{x}{1} = x \ ,$$
$$\tilde{\rho}(x) = \frac{\tilde{F}(x)}{\tilde{f}(x)} = \frac{x^{\alpha}}{\alpha x^{\alpha-1}} = \frac{x}{\alpha}$$

Consequently, $\rho(x)$ and $\tilde{\rho}(x)$ are linear in x and with Arozamena and Weinschelbaum (2009) we can follow, that the bidding behavior is unaltered if the favored bidder's value distribution is either a striclty convex or linear beta distribution.

Let $\hat{F}(x) = 1-(1-x)^{\gamma}$ be the favored bidders' value distribution, where $\gamma > 1$. According to Bagnoli and Bergstrom (2005) $\hat{F}(x)$ is logconcave and further,

$$\hat{\rho}(x) = \frac{\hat{F}(x)}{\hat{f}(x)} = \frac{1-(1-x)^{\gamma}}{\gamma(1-x)^{\gamma-1}}$$

is strictly concave. With Arozamena and Weinschelbaum (2009), we conclude that the non-favored bidder's bid is more aggressive in a Last-Call Auction than in a First-Price Auction.

□

3.2. Favoring the right bidder

Remember that conducting a Last-Call Auction with asymmetric bidders means for the auctioneer to decide which bidder is granted the ROFR. In the following we focus for the defined asymmetric bidder constellations on the question, if a selective assignment is advantageous for the auctioneer or not. For that purpose, we demonstrate that if the auctioneer knows who of the participating bidders in the Last-Call Auction is the strong and who the weak one, it might be meaningful to favor the correct bidder in order to gain a higher expected profit. That is, we consider the different expected payments in case of favoring the strong and the weak bidder. We suppose two asymmetric bidders characterized either by a convex-convex, linear-convex or concave-linear combination of value distributions. For the convex-convex and linear-convex combination the bidding behavior of the non-favored bidder remains unaltered since in both cases the price-determining bidder faces an opponent whose value distribution and density function lead to linear ratios $\rho(x)$ or $\tilde{\rho}(x)$, see Remark 3. Hence it can be shown that for a linear-convex and convex-convex combination the expected auction revenue in the Last-Call Auction is the same independent of favoring the weak or strong bidder, first. And second, that this excpected auction revenue never exceeds that in a Second-Price Auction.

Proposition 4. *Let $F_s(x) = x^\alpha$ and $F_w(x) = x$ be the bidders' value distributions, $\alpha > 1$. The auctioneer's expected profit if the weak bidder is favored $E[p_w^{LCA}]$ equals the expected payment with granting the ROFR to the strong bidder $E[p_s^{LCA}]$, $\forall \alpha > 1$, i.e.*

$$E[p_w^{LCA}] = E[p_s^{LCA}].$$

Proof. First we describe the expected payments $E[p_w^{LCA}]$ and $E[p_s^{LCA}]$ in dependence of α and then we prove that the proposition above applies for all $\alpha > 1$. In order to calculate $E[p_w^{LCA}]$, we need the strong bidder's inverse bidding strategy, because she is the price-determining bidder in this case,

$$\beta_s^{-1}(p) = p + \frac{F_w(p)}{f_w(p)} = 2p.$$

And for the bidding strategy $\beta_s(x)$ follows

$$\beta_s(x) = \frac{1}{2}x, \text{ particularly } \beta_s(1) = \frac{1}{2}.$$

So the auctioneer's expected rent if she favors the weak bidder, is

$$E[p_w^{LCA}] = \int_0^{\beta_s(1)} 1 - F_s(\beta_s^{-1}(p))dp = \int_0^{\frac{1}{2}} 1 - (2p)^\alpha dp = \frac{1}{2} - \frac{1}{2(\alpha+1)} = \frac{1}{2}\frac{\alpha}{\alpha+1}. \quad (3)$$

Under the same assumptions and granting a ROFR to the strong bidder follows for the auction revenue

$$
\begin{aligned}
E[p_s^{LCA}] &= \int_0^{\beta_w(1)} 1 - F_w(\beta_w^{-1}(p))dp = \int_0^{\frac{\alpha}{\alpha+1}} 1 - \frac{\alpha+1}{\alpha} pdp \\
&= \frac{\alpha}{\alpha+1} - \frac{1}{2}\frac{\alpha+1}{\alpha}\frac{\alpha^2}{(\alpha+1)^2} = \frac{1}{2}\frac{\alpha}{\alpha+1},
\end{aligned} \tag{4}
$$

where the inverse equilibrium bidding strategy of the weak bidder, who determines the price, is

$$
\beta_w^{-1}(p) = p + \frac{F_s(p)}{f_s(p)} = p + \frac{p^\alpha}{\alpha p^{\alpha-1}} = \frac{\alpha+1}{\alpha} p.
$$

And for the equilibrium bidding strategy $\beta_w(x)$ holds

$$
\beta_w(x) = \frac{\alpha}{\alpha+1} x, \;\; \text{particularly} \;\; \beta_2(1) = \frac{\alpha}{\alpha+1}.
$$

Comparing (3) and (4) provides the desired result. □

Thus, in the case of a weak bidder with a linear distribution and a stron bidder with a convex distribution the auctioneer's expected profit remains the same whether she favors the weak or the strong bidder, although the weak bidder submits a relatively more aggressive bid $\beta_w(x)$ for $\alpha > 1$ than the strong bidder with $\beta_s(x)$. The expected payment if the weak bidder determines the price, i.e. the strong bidder is favored, never exceeds the expected payment if the weak bidder is favored. The reason is that the weak bidder's expected valuation $E[X_w]$ is lower than the strong bidder's one $E[X_s]$ her more aggressive bidding behavior is outweighed by her weakness compared to the strong bidder, which results in equal expected profits, i.e. $E[p_w^{LCA}] = E[\beta_w(X_s)] = E[\beta_s(X_s)] = E[p_s^{LCA}]$.

Proposition 5. *Let $F_s(x) = x^{\alpha_s}$ and $F_w(x) = x^{\alpha_w}$ be the bidders' value distributions, where $1 < \alpha_w < \alpha_s$. Then the auctioneer's expected profit if the weak bidder is favored $E[p_w^{LCA}]$ equals the expected profit if she grants the ROFR to the strong bidder $E[p_s^{LCA}]$, $\forall \alpha_w, \alpha_s > 1$, i.e.*

$$
E[p_w^{LCA}] = E[p_s^{LCA}].
$$

Proof. If the weak bidder is favored the strong bidder will determine the price, where the strong bidder's inverse equilibrium bidding strategy in the convex-convex case is

$$\beta_s^{-1}(p) = p + \frac{F_w(p)}{f_w(p)} = p + \frac{p^{\alpha_w}}{\alpha_w p^{\alpha_w - 1}} = \frac{\alpha_w + 1}{\alpha_w} p.$$

This implies the strong bidder's bidding function

$$\beta_s(x) = \frac{\alpha_w}{\alpha_w + 1} x, \text{ particularly } \beta_s(1) = \frac{\alpha_w}{\alpha_w + 1}.$$

So the expected payment in a Last-Call Auction, where the weak bidder is granted a ROFR, is

$$\begin{aligned} E[p_w^{LCA}] &= \int_0^{\beta_s(1)} 1 - F_s(\beta_s^{-1}(p))dp = \int_0^{\frac{\alpha_w}{\alpha_w+1}} 1 - \left(\frac{\alpha_w + 1}{\alpha_w} p\right)^{\alpha_s} dp \\ &= \frac{\alpha_w}{\alpha_w + 1} - \left(\frac{\alpha_w + 1}{\alpha_w}\right)^{\alpha_s} \left(\frac{\alpha_w}{\alpha_w + 1}\right)^{\alpha_s + 1} \frac{1}{\alpha_s + 1} = \frac{\alpha_w}{\alpha_w + 1} \frac{\alpha_s}{\alpha_s + 1}. \end{aligned}$$

Favoring the strong bidder leads to the same inverse equilibrium bidding strategy for the weak bidder, where α_w is replaced by α_s and it holds $\beta_w(1) = \frac{\alpha_s}{\alpha_s+1}$. Therefore the auction revenue if the strong bidder is favored amounts to

$$E[p_s^{LCA}] = \int_0^{\beta_w(1)} 1 - F_w(\beta_w^{-1}(p))dp = \frac{\alpha_s}{\alpha_s + 1} \frac{\alpha_w}{\alpha_w + 1}.$$

Both expected payments $E[p_w^{LCA}]$ and $E[p_s^{LCA}]$ are symmetric in their arguments α_s and α_w and therefore correspond to each other for all $\alpha_w, \alpha_s > 1$. □

Notice that for $\alpha_s, \alpha_w \to \infty$ both bidders' bids will approach their true valuations. Further, the weak bidder's bidding strategy is more aggressive than the strong bidder's one, which is obvious, because the weak bidder faces a strong competitor, whereas the strong bidder competes against a weak one. However, the expected auction revenue by favoring the strong bidder never exceeds the expected auction revenue by favoring the weak bidder. The reason is that the more aggressive bidding behavior of the non-favored weak bidder is compensated by her lower expected valuation.

For the concave-linear combination the expected payment by favoring the weak bidder

$E[p_w^{LCA}]$ and the expected payment by favoring the strong bidder $E[p_s^{LCA}]$ differ and do not correspond to each other as seen before in the linear-convex or convex-convex combination. We find that under these assumptions favoring the weak bidder always generates a higher or equal expected revenue for the auctioneer than favoring the strong bidder.

Proposition 6. *Let $F_s(x) = x$ and $F_w(x) = 1-(1-x)^\gamma$ be the bidders' value distributions, $\gamma > 1$. Then the expected payment in a Last-Call Auction is higher or equal if the auctioneer grants the ROFR to the weak instead of the strong bidder, i.e.*

$$E[p_s^{LCA}] \leq E[p_w^{LCA}].$$

Proof. First we calculate the expected payment dependent of γ in case of favoring the weak bidder. So the inverse equilibrium bidding function of the strong and price-determining bidder is

$$\beta_s^{-1}(p) = p + \frac{F_w(p)}{f_w(p)} = p + \frac{1-(1-p)^\gamma}{\gamma(1-p)^{\gamma-1}} = p + \frac{1}{\gamma(1-p)^{\gamma-1}} - \frac{1-p}{\gamma}.$$

In order to calculate the expected auction revenue, the highest possible bid $\beta_s(1)$ is needed, which follows with

$$\begin{aligned}
&p + \frac{1}{\gamma(1-p)^{\gamma-1}} - \frac{1-p}{\gamma} = 1 \\
&\Leftrightarrow (\gamma+1)p + \frac{1}{(1-p)^{\gamma-1}} = \gamma + 1 \\
&\Leftrightarrow \frac{1}{(1-p)^\gamma} = \gamma + 1 \\
&\Leftrightarrow 1 - \sqrt[\gamma]{\frac{1}{\gamma+1}} = p \qquad \Rightarrow \beta_1(1) = 1 - \sqrt[\gamma]{\frac{1}{\gamma+1}}.
\end{aligned}$$

Then the expected payment in the Last-Call Auction with favoring the weak bidder is

$$\begin{aligned}
E[p_w^{LCA}] &= \int_0^{\beta_1(1)} 1 - F_s(\beta_s^{-1}(p))dp = \int_0^{1-\sqrt[\gamma]{\frac{1}{\gamma+1}}} (1 - p - \frac{1}{\gamma(1-p)^{\gamma-1}} + \frac{1-p}{\gamma})dp \\
&= \frac{1}{2}(1+\frac{1}{\gamma}) - \frac{1}{\gamma(2-\gamma)} + (\gamma+1)^{-\frac{2}{\gamma}} \left(-\frac{1}{2} - \frac{1}{2\gamma} + \frac{\gamma+1}{\gamma(2-\gamma)}\right). \qquad (5)
\end{aligned}$$

In order to determine the expected auction revenue with favoring the strong bidder the following inverse equilibrium bidding function of the weak and price-determining bidder is used

$$\beta_w^{-1}(p) = p + \frac{F_s(p)}{f_s(p)} = 2p.$$

Hence the weak bidder's equilibrium bidding function as well as her highest possible bid yields

$$\beta_w(x) = \frac{1}{2}x, \text{ particularly } \beta_w(1) = \frac{1}{2}.$$

So the auctioneer's expected profit is

$$\begin{aligned} E[p_s^{LCA}] &= \int_0^{\beta_w(1)} 1 - F_w(\beta_w^{-1}(p))dp = \int_0^{\frac{1}{2}} 1 - (1 - (1-2p)^\gamma)dp \\ &= \int_0^{\frac{1}{2}} (1-2p)^\gamma dp = 0 + \frac{1}{2}\frac{1}{\gamma+1} \\ &= \frac{1}{2(\gamma+1)}. \end{aligned}$$

We demand $E[p_w] - E[p_s] \geq 0$ and it follows

$$E[p_w^{LCA}] - E[p_s^{LCA}] \geq 0 \;\Leftrightarrow\; -\gamma^3 - \gamma^2 + \gamma\left((\gamma+1)^{\frac{2\gamma-2}{\gamma}} - 1\right) \begin{cases} \geq 0, & \gamma \leq 2 \\ < 0, & \gamma > 2 \end{cases}$$

For the polynom $-\gamma^3 - \gamma^2 + \gamma\left((\gamma+1)^{\frac{2\gamma-2}{\gamma}} - 1\right)$ with roots at $\gamma = 0, 1, 2$ applies

$$-\gamma^3 - \gamma^2 + \gamma\left((\gamma+1)^{\frac{2\gamma-2}{\gamma}} - 1\right) \begin{cases} \geq 0, & \gamma \leq 0 \text{ or } 1 \leq \gamma \leq 2 \\ < 0, & 0 < \gamma < 1 \text{ or } \gamma > 2 \end{cases}$$

Because of assuming that $F_w(x)$ is strictly convex only $\gamma \geq 1$ is regarded and we gain

$$E[p_w^{LCA}] - E[p_s^{LCA}] \geq 0, \quad \text{for all } \gamma \geq 1.$$

$\square$

3.3. Conditions for a-priori superiority of Last-Call Auctions

In the following, we first demonstrate that the auction revenue in a Second-Price Auction always exceeds that in a Last-Call Auction for the linear-convex and the convex-convex combination.

Proposition 7. *Let $F_s(x) = x^\alpha$ and $F_w(x) = x$ be the bidders' value distributions, $\alpha > 1$. Then the expected payment in the Second-Price Auction exceeds that in a Last-Call Auction for all $\alpha > 1$, i.e.*

$$E[p^{LCA}] < E[p^{SA}]$$

Proof. In the Second-Price Auction the bidders follow a weakly dominant bidding strategy, which signifies to bid their true valuations. This property implies that $\beta(1) = 1$ for all bidders. Hence with proposition **??** the auction revenue in the Second-Price Auction amounts to

$$\begin{aligned} E[p^{SA}] &= \int_0^{\beta(1)} 1 - F^{SA}(p)dp = \int_0^1 1 - F_s(p) - F_w(p) + F_s(p)F_w(p)dp \\ &= \int_0^1 (1 - p^\alpha - p + p^{\alpha+1})dp = \frac{1}{2} - \frac{1}{\alpha+1} + \frac{1}{\alpha+2}. \end{aligned}$$

Comparing the expected payments in the Second-Price and Last-Call Auction, which follows from Proposition 4, provides

$$\begin{aligned} E[p^{LCA}] = \frac{1}{2}\frac{\alpha}{\alpha+1} \quad &< \quad \frac{1}{2} - \frac{1}{\alpha+1} + \frac{1}{\alpha+2} = E[p^{SA}] \\ \Leftrightarrow \quad &0 < \alpha. \end{aligned}$$

□

To conclude, with an increasing $\alpha > 1$ the expected auction revenue will raise in both auction forms and converge to $\frac{1}{2}$, The reason for the higher expected payment is that one of the bidders, in this case the strong bidder, becomes stronger since α increases and therefore this strong and price-determining bidder is expected to submit an higher bid. In a Second-Price Auction the expected payment will also raise, if one of the potentially price-determining bidders becomes stronger. The fact that the expected auction revenues will never exceed $\frac{1}{2}$ in this linear-convex combination is obvious: Since we suppose that the weak bidder is favored in the Last-Call Auction the strong bidder determines the payment in dependence of the weak bidder's strength, particularly it is $\beta_s(x_s) = \frac{1}{2}x_s$. Consequently, the price-determining bid converges to $\frac{1}{2}$ because the strong bidder's expected valuation $E[X_s]$ converges to 1 for $\alpha \to \infty$. In the Second-Price Auction the second-highest bid or valuation will determine the price. If α increases the strong bidder's expected valuation converges to 1 and the weak bidder's expected valuation is $\frac{1}{2}$, which then will determine the expected payment.

Proposition 8. *Let $F_s(x) = x^{\alpha_s}$ and $F_w(x) = x^{\alpha_w}$ be the bidders' value distributions, $1 < \alpha_w < \alpha_s$. Then the expected payment in the Second-Price Auction exceeds that in the Last-Call Auction, i.e.*

$$E[p^{LCA}] < E[p^{SA}].$$

Proof. The weakly dominant bidding strategy in a Second-Price Auction is to bid one's true valuation, therefore it follows $\beta(1) = 1$ and the expected payment in the Second-Price Auction is

$$\begin{aligned} E[p^{SA}] &= \int_0^{\beta(1)} 1 - F^{SA}(p)dp = \int_0^1 (1 - p^{\alpha_s} - p^{\alpha_w} + p^{\alpha_s+\alpha_w})dp \\ &= 1 - \frac{1}{\alpha_s+1} - \frac{1}{\alpha_w+1} + \frac{1}{\alpha_s+\alpha_w+1}. \end{aligned}$$

So it follows

$$\begin{aligned} E[p^{LCA}] = \frac{\alpha_w}{\alpha_w+1}\frac{\alpha_s}{\alpha_s+1} \quad &< \quad 1 - \frac{1}{\alpha_s+1} - \frac{1}{\alpha_w+1} + \frac{1}{\alpha_s+\alpha_w+1} = E[p^{SA}] \\ &\Leftrightarrow \alpha_s + \alpha_w + 1 < (\alpha_s+1)(\alpha_w+1) \\ &\Leftrightarrow 0 < \alpha_s\alpha_w. \end{aligned}$$

which is true for all $\alpha_w, \alpha_s > 1$. □

Finally, we state that for an increasing α_s as well as for an increasing α_w the expected payment in both auction forms is augmented, where the expected payments converge to 1 for $\alpha_s, \alpha_w \to \infty$. This is immediately obvious, because both bidders become stronger, i.e. their expected valuations, $E[X_s]$ and $E[X_w]$, converge to 1 for $\alpha_s, \alpha_w \to \infty$. Notice that for $\alpha_s = 1$ or $\alpha_w = 1$ the linear-convex combination is obtained as a special case. In the following the expected auction revenues in a Last-Call Auction are compared to that in a Second-Price

Auction for the concave-linear combination. For that purpose, we first assume that the strong bidder is favored in the Last-Call Auction and find that in this case the Second-Price Auction still outperforms the Last-Call Auction in terms of expected auction revenue. Nevertheless, this result may change if the weak bidder is favored in the Last-Call Auction.

Proposition 9. *Let $F_s(x) = x$ and $F_w(x) = 1 - (1-x)^\gamma$ be the bidders' value distributions, $\gamma > 1$. Then the expected payment in a Last-Call Auction with favoring the strong bidder $E[p_s^{LCA}]$ is always lower than that in a Second-Price Auction $E[p^{SA}]$, i.e.*

$$E[p_s^{LCA}] < E[p^{SA}].$$

Proof. If the strong bidder is favored the weak bidder determines the payment through her bid. Therefore the equilibrium bidding strategy of the weak non-favored bidder is required as well as its inverse function

$$\beta_w^{-1}(p) = p + \frac{F_s(p)}{f_s(p)} = 2p \Leftrightarrow \beta_2(x) = \frac{1}{2}x.$$

With $\beta_w(1) = \frac{1}{2}$ for the expected payment in a Last-Call Auction, where the strong bidder is favored, ensues

$$\begin{aligned} E[p_s^{LCA}] &= \int_0^{\beta_w(1)} 1 - F^{LCA}(p)dp = \int_0^{\frac{1}{2}} 1 - F_w(\beta_2^{-1}(p))dp \\ &= \int_0^{\frac{1}{2}} (1 - (1 - (1-2p)^\gamma))\, dp = \int_0^{\frac{1}{2}} (1-2p)^\gamma dp \\ &= \frac{1}{2\gamma + 2}. \end{aligned} \tag{6}$$

Comparing (7) and (6) leads to

$$E[p_s^{LCA}] = \frac{1}{2\gamma+2} < \frac{1}{\gamma+2} = E[p^{SA}] \Leftrightarrow \gamma + 2 < 2\gamma + 2 \Leftrightarrow \gamma > 0,$$

which holds for all $\gamma \geq 1$. □

The next proposition will demonstrate that granting a ROFR to the weak bidder generates a higher expected auction revenue in a Last-Call Auction than in a Second-Price Auction if the parameter γ of the weak bidder's concave value distribution exceeds a certain value.

Proposition 10. *Let $F_s(x) = x$ and $F_w(x) = 1 - (1-x)^\gamma$ be the bidders' value distributions, $\gamma > 1$. Then the expected payment in the Last-Call Auction, where the weak bidder is favored, exceeds that in the Second-Price Auction if $\gamma \gtrsim 2.74509$, i.e.*

$$E[p^{SA}] < E[p_w^{LCA}], \quad \textit{for all } \gamma \gtrsim 2.74509.$$

Proof. We suppose that in the Second-Price Auction the bidders follow their weakly dominant bidding strategy and bid their true valuations, it holds $\beta(1) = 1$ and the expected payment is

$$\begin{aligned} E[p^{SA}] &= \int_0^{\beta(1)} 1 - F^{SA}(p)dp = \int_0^{\beta(1)} 1 - (1 - (1-p)^{\gamma+1})dp = \int_0^1 (1-p)^{\gamma+1}dp \\ &= \frac{1}{\gamma+2}. \end{aligned} \tag{7}$$

Assuming that the ex ante weak bidder is favored by the ROFR, the expected payment in the Last-Call Auction exceeds that in the Second-Price Auction if

$$\begin{aligned} & E[p^{SA}] < E[p_w^{LCA}] \\ & \Leftrightarrow -\gamma^4 + \gamma^3 - 2\gamma^2 + (\gamma+1)^{1-\frac{2}{\gamma}} \left(\gamma^3 + 2\gamma^2\right) \begin{cases} > 0, & \text{if } \gamma < 2 \\ \leq 0, & \text{if } \gamma \geq 2 \end{cases} \\ & \Leftrightarrow \gamma \gtrsim 2.74509. \end{aligned}$$

□

Thus a selective assignment of the ROFR to the weak bidder yields a higher expected profit for the auctioneer if a certain degree of asymmetry is given among the participating bidders, which is illustrated in figure 1.

Moreover, we raise the question, for which degree of asymmetry a randomly granted ROFR also leads to a higher expected auction revenue than the expected auction revenue in a Second-Price Auction with the same participants. In other words, the auctioneer only knows that the two participating bidders are unequally strong, but she is not informed about which bidder is the weak and which the strong one. For the selective favoritism of the weak bidder it holds

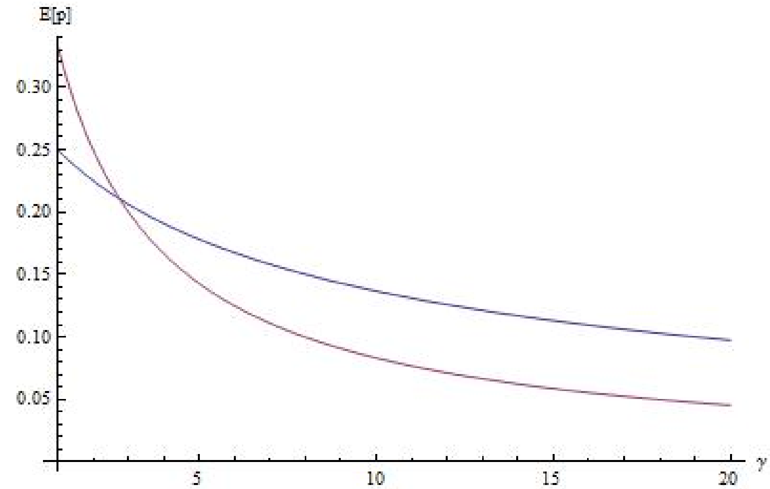

Figure 1: Concave-Linear Combination: Expected payment in a Last-Call Auction and a Second-Price Auction by increasing γ

that there exists a degree of asymmetry such that a higher expected auction revenue can be gained, which is also possible for a randomly assigned ROFR. However, favoring one of the ex ante asymmetric bidders by chance will require a higher degree of asymmetry in order to gain a higher expected auction revenue in the Last-Call than in the Second-Price Auction.

Finally, we state that a weak bidder with strictly concave beta distributed valuations entails advantageous properties for the expected auction revenue in a Last-Call Auction compared to weak bidders with linear or convex beta distributions reagrding their values.

4. Conclusion

To summarize, in the concave-linear combination it makes a difference whether the weak or the strong bidder is favored, in contrast to the other combinations. In this case we show that the auctioneer is always better off in regard to her expected profit by favoring the weak bidder. Further, besides a sufficient degree of asymmetry, the weak bidder's concave value distribution is essential for the higher expected auction revenue in a Last-Call Auction. In this case, if the ROFR is appointed selectively to the weaker bidder, the Last Call Auction generates higher expected auction revenues than a standard Second Price auction. Even if the ROFR is randomly granted to one of the two asymmetric bidders, the auction revenue in a Last-Call Auction will exceed the revenue in a Second-Price Auction as soon as the asymmetry is sufficiently large.

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