Reference-Dependent Bayesian Games

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Abstract

We present a game-theoretic model for Bayesian games that incorporates the concept of expectation-based, reference-dependent preferences of Kőszegi and Rabin (2006). We define two equilibrium concepts based on their concepts of Personal Equilibrium (PE) and Choice-Acclimating Personal Equilibrium (CPE) in which each player's reference point is endogenously determined by a consistency requirement. Although these solution concepts have already been implicitly applied, a formal definition and thorough discussion are lacking. As examples, we analyze a 2x2 game and extend the model of Lange and Ratan (2010) for sealed-bid auctions with loss averse bidders by deriving the equilibria under both concepts, PE and CPE, and show that they predict qualitatively different bidding behavior. In addition, we state several aspects of decision situations for which we hypothesize that they argue for the application of either PE or CPE.

Keywords: Reference-dependent preferences, endogenous reference point, Bayesian game, auction, loss aversion

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1. Introduction

Most economic theories build on the assumption that individuals' choices directly reflect their preferences over current or future consumption. However, a substantial body of empirical studies conducted in recent decades unambiguously suggests that actual decisions also depend on some reference level of consumption and, in particular, that decision makers are loss-averse. Several models of reference-dependent decision making have emerged, of which the most thoroughly elaborated is likely still Kahneman and Tversky's (1979; 1992) *Prospect Theory*.

One criticism of prospect theory is that it treats the reference consumption as exogenously given. Kőszegi and Rabin (2006) propose a theory of expectation-based, reference dependent preferences by assuming that choice and the formation of the reference point are inextricably linked, which allows to endogenize the reference point. Kőszegi and Rabin (2007) extend their model and propose two complementary concepts of rational decision making—*Personal Equilibrium* (PE) and *Choice-Acclimating Personal Equilibrium* (CPE)—which differ in whether the reference point is treated as fixed during the decision making process.

The present work contributes to the existing literature in two ways. In Section 3, we present a model for Bayesian games that builds on Kőszegi and Rabin's preferences. We formally introduce two types of solution concepts, one based on PE and PPE and the other on CPE. Although both concepts have already been applied in game theoretic models,[1] a general, formal definition and detailed investigation is lacking. Our approach allows to generally take reference-dependence in game theory into account and—simultaneously—use existing game-theoretic work for judging and adjusting the theory of KR's preferences. Moreover, we present two examples of Bayesian games showing that PE and CPE equilibria may even differ qualitatively and thus emphasize the importance of selecting the appropriate concept.

Before presenting our game theoretic approach, we address which of the two concepts, PE (PPE) or CPE should be applied in a particular decision situation. To our knowledge, only one distinguishing characteristic of choice situation with respect to the applicability of PE or CPE is currently mentioned in the literature, namely the length of time between the choice and consumption Kőszegi and Rabin (2007). We, however, argue that both concepts

[1] E.g. Eisenhuth (2010), Lange and Ratan (2010), Herweg et al. (2010) and Daido and Murooka (2012) for an application of CPE profiles and Ehrhart and Ott (2012) for an application of PE profiles.

certainly have their eligibility, but that the choice of the appropriate concept is more complex and depends on several aspects of the decision problem, and present some additional factors that at least seem plausible to influence the way in which the decision is made.

2. Applicability of PE (PPE) and CPE

The question which of the two personal equilibrium concepts PE (PPE) or CPE applies to a particular decision situation remains understudied. Kőszegi and Rabin (2007, p. 1059) provide a rule of thumb arguing "rather than reflecting different notions of reference-dependent utility, the two concepts are motivated by the same theory of preference, as manifested differently depending on whether the person can commit to her choice ahead of time." That is, the applicability of these solution concepts only depends on when the decision maker has to commit to her choice regarding when the relevant outcome, that is, consumption, occurs.[2]

We consider two different issues closely related to the above question. First, in the literature that applies either the PE or the CPE concept, there is little debate on the reasons why a particular concept is chosen and how it fits the reasoning given by Kőszegi and Rabin.

Second, we argue that the distinction made by Kőszegi and Rabin captures only one relevant aspect differentiating between PE and CPE decisions. There are several further facets of decision situations that should plausibly be taken into consideration. In the following, we propose some of these facets and formulate hypotheses regarding which equilibrium concept is favored by a given criterion. As stated by Kőszegi and Rabin (2007, p. 1059) "[...] because the two concepts generate distinctive behavior, observed choices can also be used to determine which concept applies." Thus, our hypotheses might improve understandings of the underlying decision processes and, in this way, be useful in finding a more reliable distinction among PE, PPE, and CPE decisions. Note that in the following discussion we not only differentiate between PE and CPE decision making but also between PE and PPE decision making. In a PE framework, the decision maker only assesses the advantage of one preselected alternative over

[2]Their intuition is as follows: If the decision maker has to commit to her choice shortly before the outcome, her beliefs over possible future outcomes relative to which the actual outcome is evaluated are unchangeable at the time of her choice. Thus, she will maximize her utility considering her expectations as given, which results in decision making according to PE. However, if a long time span passes between committing to the choice and consumption, the decision maker will anticipate that during this time her reference point will adjust to the choice she has already made. Thus, she will make her decision based on the CPE approach.

the other alternatives in the choice set by exclusively considering the preselected alternative as the reference point in the evaluation process of all choices. In a PPE framework, this procedure is carried out for all alternatives, completed by a comparison of all PEs.

- **Proximity of the alternatives** The extent to which the alternatives are similar or different may play a role. Are they, with respect to usage, comparable? As the application of PE and PPE involves the mutual comparison of alternatives related to the reference points of other alternatives, while under CPE a choice is only evaluated under its own reference point, similarity favors PE and PPE, while dissimilarity favors CPE. Different use times may be also a decisive factor. For example imagine an individual who decides whether to spend her savings either for an adventure trip around the world or to provide for her retirement. Now imagine an individual who decides whether to spend the holidays at the sea or in the mountains, or the decision when purchasing a new car of selecting between the economical or the powerful model. The dissimilarity of the alternatives in the first example, including totally different use times, argues for CPE, while the similarity of the alternatives in the latter two examples speaks for PPE.

- **Binary yes-or-no decisions (Shall I do it or not?)** Binary yes-or-no decisions may strongly favor PE decision making, particularly when the yes-decision is aimed at something the decision maker strongly wishes to have. Imagine, for example, a bidder in an English auction contemplating whether to raise her hand and thus maintain the chance to obtain the item for which she is participating in the auction or drop out of the auction and go home without the item.[3] Other examples are yes-or-no decisions regarding doing something exceptional, such as buying a house, booking an adventure holiday, or making a specific investment. As only the yes-decision brings something new and exciting, decision makers may fixate on it and fail to account for the consequences of the no-decision in equal measure. This speaks for PE decision making, taking the consequences of the yes-decision as the reference point.

- **Preconceived opinion and clear focus** Similar arguments apply to situations where the decision maker has a preconceived opinion with respect to her upcoming decision or

[3] See Ehrhart and Ott (2012).

a clear focus on one specific alternative, which also speaks for PE decision making.

- **Time to contemplate** The time available to contemplate the decision may also play a role. More allows for a more thoughtful analysis of all aspects, favoring PPE or CPE decision making, whereas decisions under time pressure instead argue for PE decision making, particularly in yes-or-no decisions or those with a clear focus.
- **Importance of the decision** The prior argument also applies to the importance of decisions, as important decisions are typically analyzed more thoroughly than less important or quick decisions.
- **Number of alternatives** A large number of alternatives may favor CPE decision making, while a small number favors the more thorough PPE decision making.

Additionally, the decision maker's characteristic disposition may also be a decisive factor. Some tend to consider all alternatives equally, while others are prone to concentrate on one distinguished alternative. Some prefer to compare alternatives mutually, while others generally separate the evaluation of different alternatives.

3. Reference-Dependent Bayesian Games

In the following section we present our game theoretic model for normal-form Bayesian games, supplemented by an example of a 2×2 game in Section 4.1 and the application to first-price and second-price sealed-bid auctions in Section 4.2.

3.1. The Basic Model

Let $I = \{1, \ldots, n\}$ the set of n players. For $i \in I$, let Θ_i be player i's type space with $\theta_i \in \Theta_i$ and $\theta := \theta_1 \times \cdots \times \theta_n \in \Theta := \Theta_1 \times \cdots \times \Theta_n$. Let $p(\theta)$ denote the common prior over all types and $p(\theta_{-i}|\theta_i)$ the conditional distribution of the other players' types for player i knowing her own type θ_i.

$A_i \subset \mathbb{R}$ denotes player i's non-empty action space with $a_i \in A_i$ and $a := a_1 \times \cdots \times a_n \in A := A_1 \times \cdots \times A_n$. A pure strategy for player i is a measurable mapping $s_i : \Theta_i \to A_i$ prescribing an action for each possible type, i.e. $a_i := s_i(\theta_i)$, and a mixed strategy is a probability distribution over her pure strategies. Let S_i denote player i's strategy set and $S = S_1 \times \cdots \times S_n$ the set of all strategies.

$C_i := C_i^1 \times \cdots \times C_i^K \in \mathbb{R}^K$ denotes player i's set of all possible K-dimensional consumption bundles with $c_i \in C_i$ and $h_i : A \to C_i$ denotes the outcome function that assigns a consumption bundle to every action vector. Thus, for any type vector $\theta \in \Theta$, any consumption bundle for player i is a deterministic result of the chosen strategy profile, that is $c_i := h_i(a) = h_i(s(\theta))$.

Players' preferences are defined according to Kőszegi and Rabin (2006). For each consumption dimension $k \in \{1, \ldots, K\}$, $m_i^k : C_i^k \times \Theta \to \mathbb{R}$ denotes player i's continuous and increasing consumption utility function and $\mu_i^k : \mathbb{R} \to \mathbb{R}$ her gain-loss utility.[4] Note that this formulation allows to model interdependent valuations. The reference-dependent utility from direct consumption of the bundle $c_i \in C_i$ relative to a reference bundle $r_i \in C_i$ is assumed to be additively separable across all K relevant dimensions of consumption, that is,

$$u_i(c_i, \theta | r_i, \theta') = \sum_{k=1}^{K} \left[m_i^k(c_i^k, \theta) + \mu_i^k \left(m_i^k(c_i^k, \theta) - m_i^k(r_i^k, \theta') \right) \right].$$

This model allows for uncertainty over future consumption and stochastic reference points. Player i's utility derived from a probability distribution F over future outcomes c_i relative to a probability distribution G over reference bundles r_i is given by

$$U_i(F, \theta | G, \theta') = \iint_{C_i} u_i(c_i, \theta | r_i, \theta'_i) dF(c_i) dG(r_i).$$

This is assumed to be common knowledge.

The following definition summarizes the notion of Reference-Dependent Bayesian Games (RD-Bayesian Games).

[4] To be more precise, Kőszegi and Rabin assume that μ satisfies the following assumptions first stated by Bowman et al. (1999):

A0 (Regularity)	$\mu(0) = 0$, and μ is twice differentiable on $\mathbb{R} \backslash \{0\}$.
A1 (Preference Monotonicity)	μ is strictly increasing.
A2 (Small Stake Loss Aversion)	$\mu'_+(0) > 0$ and $\lambda := \frac{\mu'_-(0)}{\mu'_+(0)} > 1$.
A3 (Large Stake Loss Aversion)	$x > x' > 0 \Rightarrow \mu(x) + \mu(-x) < \mu(x') + \mu(-x')$.
A4 (Diminishing Sensitivity)	$\mu''(x) \leq 0$ for $x > 0$ and $\mu''(x) \geq 0$ for $x < 0$.

.

Definition 1. *A* RD-Bayesian game G *is the tuple*

$$G = \{I, \Theta, S, A, h, C, m, \mu\},$$

where I is the set of all players, Θ is the set of all types, S is the players' strategy set and A their actions set, $h := (h_1, \ldots, h_n)$ is the vector of assignment functions, $C := C_1, \ldots, C_n$ is the set of players' achievable consumption bundles, $m := (m_1, \ldots, m_n)$ is the players' consumption utility, and $\mu := (\mu_1, \ldots, \mu_n)$ is their gain-loss utility.

As the Kőszegi-Rabin utilities, possible types, and the prior probability distribution are common knowledge, any strategy profile $s = (s_1, \ldots, s_n)$ results for player i in a lottery over her possible consumption bundles $c_i \in C_i$, which reflects her uncertainty over the other players' types. To formalize this, let $F_i(\cdot|s, \theta_i)$ denote the probability distribution over consumption bundles $c_i = h_i(s(\theta)) \in C_i$ resulting from the strategy profile s and the conditional distribution of types $p(\theta_{-i}|\theta_i)$. Now, we can define for any player i with type θ_i her reference-dependent utility from a strategy profile $s \in S$ relative to a strategy profile $t \in S$ as the Kőszegi-Rabin utility derived from the lottery $F_i(\cdot|s, \theta_i)$ relative to the lottery $F_i(\cdot|t, \theta_i)$, that is

$$U_i^{\theta_i}(s|t) = \iint_{\Theta_{-i}} u_i(c_i, (\theta_i, \theta_{-i})|r_i, (\theta_i, \theta'_{-i}))dp(\theta_{-i}|\theta_i)dp(\theta'_{-i}|\theta_i) \tag{1}$$

where

$$c_i = h_i(s(\theta_i, \theta_{-i})) \in C_i \text{ and } r_i = h_i(t(\theta_i, \theta'_{-i})) \in C_i \, .$$

Next, we formalize the solution concept for RD-Bayesian games that builds on the notion of personal equilibrium as defined in Kőszegi and Rabin (2007). In essence, our framework requires that each player's strategy choice constitutes—taken the other players' strategies as given—a personal equilibrium. The timing of events assumed in our model, which is crucial for the following equilibrium concepts, is illustrated in Figure 1.

First, each player $i \in I$ learns her own type and updates her beliefs about the other players' types to $p(\theta_{-i}|\theta_i)$ according to Bayes' rule. Then, she forms her beliefs about the other players' strategies, that is, she formulates her subjective probabilities of how likely any strategy profile $s_{-i} \in S_{-i}$ is. Next, player i forms her beliefs about her own choice of strategy and thereby her beliefs about her expected payoffs. These expectations serve as the stochastic reference

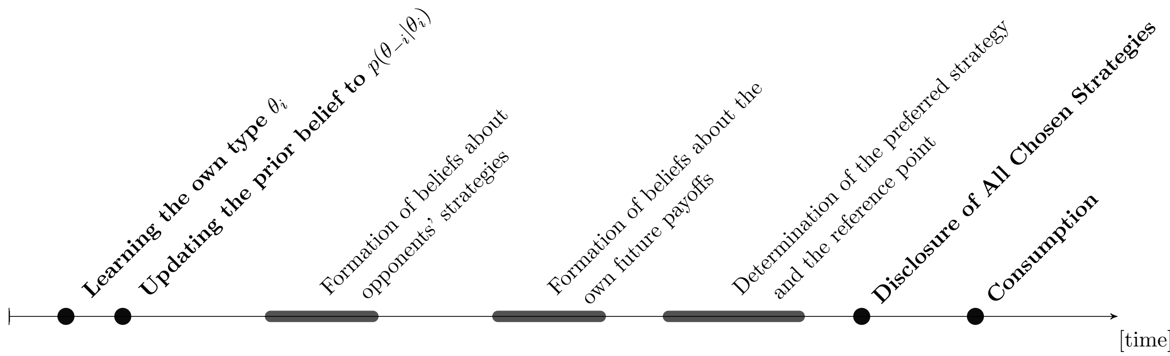

Figure 1: Time line illustrating the sequence of relevant events for a particular player in a RD Bayesian game. Events that last for a period of time are shown as gray bars, one-shot events as dots. External events are labeled in bold.

point to which each strategy is compared. As in Kőszegi and Rabin (2006), we assume that the players' preferences depend on *lagged* expectations rather than expectations at the time of choice. Kőszegi and Rabin argue that "this does not assume that beliefs are slow to adjust to new information or that people are unaware of the choices that they have just made—but that preferences do not instantaneously change when beliefs do." Briefly after the formation of her reference point, she determines her preferred strategy, and all players simultaneously disclose their choices. Then, a period of time passes until they receive their payoff dependent on the strategy profile chosen by all players. It bears emphasizing that the time line in Figure 1 is only schematic and serves as illustration of the chronological order of the relevant events.

Following Kőszegi and Rabin (2007), we define two complementary types of solution concepts that should be applied depending on various psychological factors.[5] Both concepts require optimal choice and the consistency of the expectations and choices of all players.

3.2. Personal Equilibrium Profiles

Our first two equilibrium concepts are based on the Kőszegi and Rabin (2006) notion of personal equilibrium (PE).[6]

Definition 2. *Let G be an RD-Bayesian game.*
A strategy profile $s = (s_1, \dots, s_n) \in S$ is a Personal Equilibrium (PE) profile of G, if for each

[5] See Section 2.

[6] Note that, for reasons of clarity, in Kőszegi and Rabin (2007) this concept is called choice-unacclimating personal equilibrium (UPE).

player $i \in I$ and each type $\theta_i \in \Theta_i$ it is

$$U_i^{\theta_i}\big(s|s\big) \geq U_i^{\theta_i}\big((s_i', s_{-i})|s\big) \qquad \textit{for each } s_i' \in S_i.$$

That is, the strategy profile $s = (s_1, \ldots, s_n)$ is a PE profile if and only if it constitutes for each player a personal equilibrium, as defined in Kőszegi and Rabin (2006). Definition 2 can be interpreted as follows. If player i anticipates that the other players will choose strategies according to the strategy vector s_{-i} and she will choose s_i, then in equilibrium, choosing the strategy s_i should be optimal in the sense of Kőszegi and Rabin's personal equilibrium.

From a descriptive perspective, it is likely that the players, after having formed their beliefs about the other players' strategy choices are, when contemplating their own choice, able to anticipate the possible equilibrium payoffs, compare them, and finally select the PE profile yielding the highest ex ante overall utility.[7] We formalize this descriptive intuition in the following definition.

Definition 3. *Let G be a RD-Bayesian game.*
A strategy profile $s = (s_1, \ldots, s_n) \in S$ is a Preferred Personal Equilibrium (PPE) profile of G, if s is a PE profile and if for each player $i \in I$ and each type $\theta_i \in \Theta_i$ it

$$U_i^{\theta_i}\big(s|s\big) \geq U_i^{\theta_i}\big((s_i', s_{-i})|(s_i', s_{-i})\big)$$

for all strategy profiles $s_i' \in S_i$, such that (s_i', s_{-i}) is a PE profile.

Note that Definition 3 does not require that the strategy s_i yields for player i the overall highest utility she can achieve in a PE profile. It only requires that, provided the other players' strategy choices are fixed, s_i yields her best PE profile.

In games with ex ante symmetric players—that is, games with a symmetric common prior in which all players have identical type spaces, consumption sets, payoff and utility functions—it will often seem behaviorally convenient to focus on symmetric equilibria.

[7]See also the argument in (Kőszegi and Rabin, 2007, p. 1056).

3.3. Choice-Acclimating Equilibrium Profiles

Next, we turn to the second type of equilibrium profiles based on Kőszegi and Rabin's (2007) notion of choice-acclimating personal equilibrium.

Definition 4. *Let G be a RD-Bayesian game.*
A strategy profile $s = (s_1, \ldots, s_n) \in S$ is a Choice-Acclimating Personal Equilibrium (CPE) profile of G, if for each player $i \in I$ and each type $\theta_i \in \Theta_i$ it is

$$U_i^{\theta_i}\big(s|s\big) \geq U_i^{\theta_i}\big((s'_i, s_{-i})|(s'_i, s_{-i})\big) \qquad \textit{for each } s'_i \in S_i.$$

Thus, the strategy profile $(s_1, \ldots, s_n)$ is a CPE profile if and only if it constitutes a choice-acclimating personal equilibrium (CPE) for each player, as defined in Kőszegi and Rabin (2007). Note that the only difference between Definition 4 and Definition 2 lies in the way the unilateral deviation is evaluated, namely, in what the reference point in this evaluation is. Whereas in PE profiles all players' reference strategies, and thus, their expectations over their own future consumption are taken as given, in CPE profiles their reference points always adjust to the considered strategy.

3.4. Results and Remarks

The strategic intuition behind both types of equilibria we propose resembles the intuition behind standard Bayes-Nash equilibria: unilateral deviations from equilibrium profiles cause a (weak) decline in the deviating player's expected utility. However, the presented equilibrium concepts should not be reduced to Bayes-Nash equilibria based on a more refined utility theory, as they are not only models of behavior but also models of the formation of expectations in strategic decision situations. Kőszegi and Rabin's framework of endogenous, expectation-based reference point determination adds a new, psychological dimension to Bayesian games.

However, as the utility defined in (1) is continuous in the strategies, many major results from classical Bayesian Game Theory bearing on that argument can easily be transferred to RD-Bayesian games. We will state two of these results explicitly, as we will provide examples of such games in the following. For instance, we have that for every RD-Bayesian Game, both PE and CPE profiles in mixed strategies exist if the action and type sets are finite.

Proposition 1. *Let G be a RD-Bayesian Game with finite action space A and finite type space $\Theta = \Theta_1 \times \Theta_2 \times \cdots \times \Theta_n$. Then, there exist both PE and CPE profiles in mixed strategies.*

For the proof see the Appendix.

If the action and type sets are continuous, the proposition also holds for equilibria in pure strategies under some additional conditions. For example, Maskin and Riley's (2000) result for static auctions also holds under similar conditions for bidders with KR-preferences.[8]

Proposition 2. *Let G be a reference-dependent auction game, that is, a standard private values first-price or second-price auction in which all players have Kőszegi-Rabin utilities as defined in equation (1), with continuous type space $\Theta = \Theta_1 \times \Theta_2 \times \cdots \times \Theta_n$. Then, if the types are distributed independently, there exist both PE and CPE profiles in pure strategies.*

For the proof, see the Appendix.

4. Examples and Applications

In the following, we consider two applications of RD-Bayesian games. First, we present a symmetric Bayesian game in normal form. Second, we extend the sealed-bid auction model of Lange and Ratan (2010) to PE and PPE profiles.

4.1. A Symmetric 2x2 Game

Consider the symmetric Bayesian two-player game in Figure 2.

		Player 2	
		X	Y
Player 1	X	a,a	b,0
	Y	0,b	θ_1, θ_2

Figure 2: RD-Bayesian game with $c > a > b > 0$ and $\theta_1 = \theta_2 = \{0 : p; c : 1 - p\}$

.

Both players have two pure strategies X and Y and their types θ_1 and θ_2 are independently drawn from $\{0, c\}$ with probability $p \in [0, 1]$ for type 0 and $1 - p$ for type c. For simplicity and illustration purposes, we only consider symmetric equilibria in pure strategies.

[8]For an example see section 4.2.

If player 1 is of type 0:

		Player 2	
		X	Y
Player 1	X	a,a	b,0
	Y	0,b	$0, \theta_2$

If player 1 is of type c:

		Player 2	
		X	Y
Player 1	X	a,a	b,0
	Y	0,b	c, θ_2

Figure 3: RD-Bayesian-Game with $c > a > b > 0$ and $\theta_2 = \{0 : p; c : 1-p\}$ for both possible types of player 1.

To understand the basic properties of this game, we first apply standard game theory. If a player is of type 0, strategy X is strictly dominant. Thus, if both players are of type 0, there exists a unique Nash equilibrium, in which both players choose X. If both players are of type c, the game has two symmetric Nash equilibria in pure strategies: both players play X or both players play Y (Pareto dominant equilibrium).

Let (σ_0, σ_c) denote a player's strategy in the Bayesian game, i.e. the player plays $\sigma_0 \in \{X, Y\}$ if she is of type 0 and she plays $\sigma_c \in \{X, Y\}$ if she is of type c. The game has either one or two symmetric Bayesian Nash equilibria (BNE), depending on p:

i. (X, X) constitutes a symmetric BNE for $p \in [0, 1]$.

ii. (X, Y) constitutes a symmetric BNE for $0 \leq p \leq \frac{c-b}{a+c-b}$

Therefore, (X, X) constitutes the unique symmetric BNE for $p > \frac{c-b}{a+c-b}$.

Now we apply our framework of RD-Bayesian games. We assume that the KR utility of both players is identical. The consumption utility m is given by $m(x) = x$ and the gain-loss utility μ is piecewise linear,

$$\mu(x) = \begin{cases} \gamma x & \text{if } x \geq 0, \\ \lambda x & \text{if } x < 0, \end{cases}$$

with $\lambda \geq \gamma \geq 0$.[9]

For player $i \in \{1, 2\}$ with type $\theta_i \in \{0, c\}$ we calculate the reference-dependent utility $U_i^{\theta_i}(\sigma|\sigma')$ of a strategy $\sigma \in \{X, Y\}$ given the reference strategy $\sigma' \in \{X, Y\}$. Player i's beliefs about the other player $-i$'s strategy choice are given by $q \in [0, 1]$: q is the probability that

[9]Note the difference from the notation used in most related papers (e.g., Kőszegi and Rabin, 2006), where $\mu(x) = \eta x$ for $x > 0$ and $\mu(x) = \eta\lambda(x)$ for $x \leq 0$ with $\eta \geq 0$ and $\lambda \geq 1$, such that η corresponds to our gain parameter γ and $\eta\lambda$ corresponds to our loss parameter λ.

$-i$ chooses X and, thus, $1-q$ is the probability that $-i$ chooses Y.

$$U_i^0(X|X) = qa + (1-q)b - (1-q)q(\lambda - \gamma)(a-b) \tag{2}$$

$$U_i^0(X|Y) = qa + (1-q)b + \gamma[qa + (1-q)b] \tag{3}$$

$$U_i^0(Y|X) = -\lambda[qa + (1-q)b] \tag{4}$$

$$U_i^0(Y|Y) = 0 \tag{5}$$

$$U_i^c(X|X) = qa + (1-q)b - (1-q)q(\lambda - \gamma)(a-b) \tag{6}$$

$$U_i^c(X|Y) = qa + (1-q)b + q\gamma[qa + (1-q)b)] - (1-q)\lambda[q(c-a) + (1-q)(c-b)] \tag{7}$$

$$U_i^c(Y|X) = (1-q)c + (1-q)\gamma[q(c-a) + (1-q(c-b))] - q\lambda[qa + (1-q)b] \tag{8}$$

$$U_i^c(Y|Y) = (1-q)c - q(1-q)(\lambda - \gamma)c \tag{9}$$

Simple computation yields that the following inequalities can be fulfilled for certain values of $c > a > b > 0$ and $\lambda > \gamma \geq 0$, which for certain beliefs induce the following symmetric equilibrium profiles:

$$\begin{array}{llllll}
(2) \geq (4) & \text{and} & (6) \geq (8) & \text{with} & q = 1 & \Rightarrow & (X,X) \text{ is a symmetric PE profile} \\
(2) \geq (5) & \text{and} & (6) \geq (9) & \text{with} & q = 1 & \Rightarrow & (X,X) \text{ is a symmetric CPE profile} \\
(2) \geq (4) & \text{and} & (9) \geq (7) & \text{with} & q = p & \Rightarrow & (X,Y) \text{ is a symmetric PE profile} \\
(2) \geq (5) & \text{and} & (9) \geq (6) & \text{with} & q = p & \Rightarrow & (X,Y) \text{ is a symmetric CPE profile} \\
(5) \geq (2) & \text{and} & (6) \geq (9) & \text{with} & q = 1-p & \Rightarrow & (Y,X) \text{ is a symmetric CPE profile}
\end{array}$$

Note that the strategy (X,X) always constitutes a symmetric PE profile and a CPE profile, whereas the strategies (X,Y) and (Y,X) constitute PE and CPE profiles, respectively, for only some parameter constellations.

Similar arithmetics show that the strategy (Y,X) cannot be a PE profile and (Y,Y) cannot be either a PE nor a CPE profile for any parameters $c > a > b > 0$ and $\lambda > \gamma \geq 0$.

To illustrate these results, let $a = 40, b = 5, c = 50, \lambda = 2.6, \gamma = 0$ (see Figure 4).

Using the inequalities (2) – (9), we calculate the ranges of p in which symmetric PE/PPE and CPE profiles exist (Table 1). As stated above, (X,X) constitutes a symmetric PE/PPE and a symmetric CPE profile for all $p \in [0,1]$. (X,Y) only is an a symmetric PE/PPE and

	Player 2	
	X	Y
Player 1 X	$40, 40$	$5, 0$
Y	$0, 5$	θ_1, θ_2

$$\theta_1 = \theta_2 = \{0 : p; 50 : 1 - p\}$$

Figure 4: RD-Bayesian game with $a = 40, b = 5, c = 50$.

	BNE	PE	PPE	CPE
(X,X)	$p \in [0, 1]$	$p \in [0, 1]$	$p \in [0, 1]$	$p \in [0, 1]$
(X,Y)	$p \lessapprox 0.53$	$p \lessapprox 0.53$	$p \lessapprox 0.48$	$p \lessapprox 0.11$
(Y,X)	–	–	–	$0.49 \lessapprox p \lessapprox 0.58$
(Y,Y)	–	–	–	–

Table 1: Equilibria of the RD-Bayesian-Game with $a = 40, b = 5, c = 50, \lambda = 2.6, \gamma = 0$.

CPE if p does not exceed a certain bound, i.e. the probability of being of type 50 must be sufficiently high. The exact value p^* of the bound depends on the concept, and we have the following ranking: $p^*_{BNE} = p^*_{PE} > p^*_{PPE} > p^*_{CPE}$. Thus, if strategy (X, Y) constitutes a symmetric CPE profile, it is also a PPE profile and if it is a PPE profile it also is a PE profile and a BNE. However, not every PE profile is also a PPE profile (e.g. $p = 0.5$).

The intuitively unappealing strategy (Y, X) in which a player of type 0 chooses the dominated strategy Y, while she opts for X if she is of type 50, can constitute a CPE profile for p around 0.5, but not a PE or PPE profile. This refers to the fact that the CPE concept permits the selection of stochastically dominated lotteries.

4.2. Application to Sealed Bid Auctions

We consider first- and second-price auctions with independent private values and loss averse bidders. In essence, we follow the approach of Lange and Ratan (2010) with two variations. First, we also take the sensation of unexpected gains of the good into account, but neglect loss and gain sensation in money. Second, in addition to the CPE profiles, we also calculate the PE and PPE profiles in the sense of Section 3.1.

There are n loss-averse bidders that have Kőszegi-Rabin preferences and treat money and the auctioned good as two different dimensions of consumption. The bidders' possible consumption bundles are pairs $c = (c_g, c_m)$ consisting of the auctioned good $c_g \in \{0, 1\}$ and of money c_m, which we treat as a numeraire. Thus, for deterministic levels of consumption, each bidder's utility, which we assume to be additively separable across dimensions, is

$$u(c|r) = u^g(c_g|r_g) + u^m(c_m|r_m), \tag{10}$$

where u^g denotes the utility from consuming c_g units of the good when expecting the consumption of r_g units, and u^m denotes the utility from c_m units of money when expecting r_m monetary units. For each dimension $k \in \{g, m\}$, we assume that each bidders' utility can be decomposed into a consumption utility $m^k(c_k)$ and a gain-loss utility $\mu^k(\cdot)$,

$$u^k(c_d|r_d) = m^k(c_d) + \mu^k\big(m^k(c_d) - m^k(r_d)\big). \tag{11}$$

We assume $m^m(c_m) = c_m$ and $\mu^m(x) = 0$, i.e. in the domain of money, bidders are risk neutral and exhibit no loss aversion. In the domain of the good, we define $m^g(0) = 0$ and $m^g(1) = v$, where v denotes the bidder's private intrinsic valuation of the good. Each bidder's valuation is private information and an independent realization of a probability distribution F over the interval $[0, \overline{v}]$ with full support. From the perspective of any particular bidder, let G denote the distribution of the highest valuation of the other bidders, i. e. $G(v) = F^{n-1}(v),\ v \in [0, \overline{v}]$. Furthermore, bidders are loss-averse in the domain of the good and we assume the gain-loss utility to be linear both in the domain of gains and losses,

$$\mu^g(c_g|r_g) = \begin{cases} \gamma v & \text{if } c_g = 1, r_g = 0, \\ 0 & \text{if } c_g = r_g, \\ -\lambda v & \text{if } c_g = 0, r_g = 1, \end{cases} \tag{12}$$

with $\lambda \geq \gamma \geq 0$.[10] The parameters λ and γ can be regarded as measures of aversion to unexpected losses and pleasure derived from unexpected gains, respectively.

For a probability distribution of consumption bundles C and a probability distribution of

[10] We use the same notation as in Section 4.1, which differs from the notation used in most related papers. Our notation is more similar to that of Lange and Ratan (2010), who assume $\gamma = 0$.

reference bundles R, a bidder's utility is given by her expected utility over all combinations of risky outcomes and reference points:

$$U(C|R) = \iint u(c|r)dC(c)dR(r). \tag{13}$$

4.2.1. First-price auction

If bidder i submits a bid $b \geq 0$ in the first-price sealed-bid auction, she will consume the bundle $(1, -b)$ if she wins the auction and $(0, 0)$ if not. From the perspective of bidder i, let $H(b)$ denote the probability, that she will win the auction with the bid $b \geq 0$. Thus, each bidder i faces the decision problem of choosing one lottery out of the set

$$D = \left\{ \big((1, -b) : H(b),\ (0, 0) : 1 - H(b) \big) :\ b \geq 0 \right\}. \tag{14}$$

According to (13), bidder i's utility resulting from any bid $q \geq 0$ when her reference point is determined by the bid $b \geq 0$ is given by

$$U(q|b) = (v - q)H(q) - \lambda v H(b)(1 - H(q)) + \gamma v(1 - H(b))H(q). \tag{15}$$

Bid b constitutes a PE for bidder i if and only if $U(b|b) \geq U(q|b)$ for all $q \geq 0$ and a CPE if and only if $U(b|b) \geq U(q|q)$ for all $q \geq 0$.

We are interested in symmetric PE and CPE profiles, represented by equilibrium bidding functions $\beta : [0, \overline{v}] \to \mathbb{R}_0^+$.

Proposition 3. *In the first price auction, the bidding function*

$$\beta^{FA,PE}(v) = \frac{\int_0^v s\big[1 + \lambda G(s) + \gamma(1 - G(s))\big]g(s)ds}{G(v)}, \tag{16}$$

where G and g denote the distribution and the density of the other bidders' highest intrinsic valuation, respectively, constitutes the unique symmetric PE profile and also the unique symmetric PPE profile.

For the proof, see the Appendix.

That is, in the unique symmetric PE and PPE profile, a bidder with $v > 0$ submits a bid that is higher than the risk-neutral equilibrium bid $\beta^{FA}(v) = \frac{\int_0^v sg(s)ds}{G(v)}$ when bidders are expected-utility maximizers, which is included as a special case for $\lambda = \gamma = 0$. Furthermore, the bid strictly increases in both λ and γ. Note, however, that $\beta^{FA,PE}$ does not constitute a CPE profile. To see this, suppose that all other bidders $j \neq i$ bid $\beta^{FA,PE}(v_j)$ and bidder i bids $\beta^{FA,PE}(x)$. Then, bidder i's utility is given by (with $\beta := \beta^{FA,PE}$)

$$U(\beta(x)|\beta(x)) = (v - \beta(x))G(x) - \lambda v G(x)(1 - G(x)) + \gamma v(1 - G(x))G(x)$$

with the derivative

$$\begin{aligned}\frac{dU(\beta(x)|\beta(x))}{dx} &= (v\,(1 + (\lambda - \gamma)(2G(x) - 1)]) - \beta(x))\,g(x)) - G(x)\beta'(x) \\ &= ((v - x)(1 + (\lambda - \gamma)G(x)) - v(\lambda - \gamma)(1 - G(x)) - \gamma x)\,g(x),\end{aligned}$$

where we use $\beta'(x) = [x\,(1 + \gamma + (\lambda - \gamma)G(x)) - \beta(x)]\,\frac{g(x)}{G(x)}$. Obviously, $\frac{dU(\beta(x)|\beta(x))}{dx} < 0$ for all $x \leq v$. Thus, it is not optimal for player i to bid $\beta(v)$ and the best response $\beta(x)$ in the CPE sense is strictly smaller than $\beta(v)$.[11]

For improved comparability, we state Lange and Ratan's (2010) result characterizing the CPE profiles including the additional parameter γ.

Proposition 4. *In the first-price auction, the bidding function*

$$\beta^{FA,CPE}(v) = \max\left\{\frac{\int_0^v s\big(1 + (\lambda - \gamma)(2G(s) - 1)\big)g(s)ds}{G(v)},\ 0\right\}, \tag{17}$$

where G and g denote the distribution and the density of the other bidders' highest intrinsic valuation, respectively, constitutes the unique symmetric CPE profile.

For the proof, see the Appendix.

As noted by Lange and Ratan and contrary to the PE profile, in the CPE profile only bidders with a valuation above a threshold will submit a positive bid. Moreover, λ and γ do not have an unambiguous effect on $\beta^{FA,CPE}$: for valuations v with $G(v) < \frac{1}{2}$, $\beta^{FA,CPE}$ is increasing in γ and decreasing in λ and vice versa for v with $G(v) > \frac{1}{2}$.

The comparison of the symmetric PE and CPE profiles yields the following result.

[11] Note, however, that $\beta(x)$ is not the best response for player i in the PE sense as is shown in the proof of Proposition 3.

Corollary 1. *In the first-price auction, the bid of a bidder with positive valuation $v > 0$ and $\lambda > \gamma \geq 0$ in the unique symmetric PE profile is strictly higher than in the unique symmetric CPE profile,*

$$\beta^{FA,PE}(v) > \beta^{FA,CPE}(v) \quad \textit{for all } v \in (0, \overline{v}].$$

For the proof see the Appendix.

4.2.2. Second-price Auction

If bidder i submits a bid $b \geq 0$ in the second-price sealed-bid auction, she will consume the bundle $(1, -p)$ if she wins the auction at price p and $(0, 0)$ if not. Let $H(b)$ denote the probability that i will win the auction with the bid b. Thus, bidder i's utility resulting from the bid $q \geq 0$ when her reference point is determined by the bid b is given by

$$U(q|b) = \int_0^q (v - p)h(p)dp - \lambda v H(b)(1 - H(q)) + \gamma v(1 - H(b))H(q). \tag{18}$$

Again, bid b constitutes a PE for bidder i if and only if $U(b|b) \geq U(q|b)$ for all $q \geq 0$ and a CPE if and only if $U(b|b) \geq U(q|q)$ for all $q \geq 0$.

As in the first-price auction, there exists a unique symmetric PE and PPE profile.

Proposition 5. *In the second price auction, the bidding function*

$$\beta^{SA,PE}(v) = v\big(1 + \lambda G(v) + \gamma(1 - G(v))\big), \tag{19}$$

where G and g denote the distribution and the density of the other bidders' highest intrinsic valuation, respectively, constitutes the unique symmetric PE profile and also the unique symmetric PPE profile.

For the proof, see the Appendix.

That is, in the unique PE and PPE profile, each bidder with $v > 0$ submits a positive bid that exceeds v and strictly increases in both λ and γ. If $\lambda = \gamma = 0$ the bidding function $\beta^{SA,PE}$ simplifies to the dominant strategy $\beta^{SA}(v) = v$. Note that for $\lambda > 0$ and/or $\gamma > 0$, $\beta^{SA,PE}$ is not a dominant strategy. However, it is obvious from (19) that $\beta^{SA,PE}(v)$ lies in the interval $[v(1 + \gamma), v(1 + \lambda)]$, regardless of the other bidders' bids.

The unique symmetric PPE profile does not constitute a CPE profile. As before, suppose that all other bidders $j \neq i$ bid $\beta^{SA,PE}(v_j)$ and bidder i bids $\beta^{SA,PE}(x)$. Then, bidder i's utility is given by (with $\beta := \beta^{SA,PE}$)

$$U(\beta(x)|\beta(x)) = \int_0^{\beta(x)} (v-p)h(p)dp - \lambda v G(x)(1-G(x)) + \gamma v(1-G(x))G(x)$$

with the derivative

$$\frac{dU(\beta(x)|\beta(x))}{dx} = \big(v\big(1+(\lambda-\gamma)(2G(x)-1)\big) - \beta(x)\big)\, h(\beta(x))\beta'(x).$$

By using $\beta'(x) > 0$, $h(\beta(x)) > 0$ and inserting the definition of $\beta(x)$, it follows that

$$\frac{dU(\beta(x)|\beta(x))}{dx} < 0 \quad \text{for all } x \geq v.$$

Thus, from the CPE perspective, it is not optimal for player i to bid $\beta(v)$. The best response $\beta(x)$ in the CPE sense is strictly smaller than $\beta(v)$.

Again, we replicate Lange and Ratan's result using our utility function.

Proposition 6. *In the second price auction, the bidding function*

$$\beta^{SA,CPE}(v) = \max\left\{v\big(1+(\lambda-\gamma)(2G(v)-1)\big), 0\right\}, \tag{20}$$

where G and g denote the distribution and the density of the other bidders' highest valuation, respectively, constitutes the unique symmetric CPE profile.

For the proof, see the Appendix.

Similar to the first-price auction, in the CPE profile only bidders with a valuation above a threshold will submit a positive bid, and, again, the parameters λ and γ do not have an unambiguous effect: for v with $G(v) < \frac{1}{2}$, $\beta^{FA,CPE}$ is increasing in γ and decreasing in λ and vice versa for v with $G(v) > \frac{1}{2}$. Moreover, a bidder with v such that $G(v) < \frac{1}{2}$ submits a bid that is lower than v to protect herself from making unreasonable expectations.

The comparison of the symmetric PE and CPE profiles yields the following result.

Corollary 2. *In the second-price auction, the bid of a bidder with positive valuation $v > 0$*

in the unique symmetric PE profile is not lower than in the unique symmetric CPE profile,

$$\beta^{SA,PE}(v) \geq \beta^{SA,CPE}(v) \quad \textit{for all } v \in (0, \overline{v}],$$

and equality only holds for $v = \overline{v}$ and $\gamma = 0$.

For the proof see Appendix.

4.3. Discussion

The examples of the normal-form game and the sealed-bid auctions reveal that and how the PE, PPE, and CPE equilibrium profiles may differ from each other and from the traditional Bayes-Nash equilibrium.

In the presented normal-form game, PE profiles always coincide with the classical Bayes-Nash equilibria, whereas the PPE and CPE profiles differ from the Bayes-Nash equilibria. In particular, the CPE concept may seem behaviorally implausible.[12]

In the sealed-bid auction example, both PE/PPE and CPE profiles differ from the Bayes Nash equilibrium. In this way, a reference-dependent model of auctions with either of these solution concepts can account for the frequently observed overbidding in real-world auctions and provide an explanation for the notions of *joy of winning* and *fear of losing*, which is solely based on a utility function that incorporates loss aversion. However, despite these similarities, the bidding functions derived in the symmetric PE/PPE and CPE profiles lead to behaviorally different predictions for human behavior. Not only are the bids in the PE profiles (substantially) higher than in the CPE profile, but the CPE profile also predicts that bidders with small valuations might strongly shade their bids or even not bid at all, whereas the PE profile always predicts higher bids than the risk-neutral Bayes-Nash equilibrium. Furthermore, whereas in the PE profile both parameters λ and γ in the good dimension have a positive effect on the bids, this does not hold for the CPE profile.

These examples illustrate the problem that models that apply Köszegi and Rabin's framework of endogenous reference formation automatically face: without the correct psychological

[12]This is because the CPE concept allows the choice of first-order stochastically dominated lotteries. See also Kőszegi and Rabin (2007).

judgment of which equilibrium concept applies, the models' predictions may not only be inexact but may even point in the wrong direction. We conjecture that in the our normal-form game example, the PE/PPE profiles are behaviorally more appealing, as with only two actions available, the decision maker is likely to compare the two alternative actions in a pairwise fashion (see Section 2). In the case of the sealed-bid auctions, the reasoning is more complex and there are arguments supporting both PE/PPE and CPE decision making. Clearly, there is a need for an experimental investigation of the applicability of the two coexisting equilibrium concepts, and games where these concepts predict different behavior, as presented in this section, might prove to be a starting point for such an examination.

5. Conclusion

We apply the framework of Köszegi and Rabin (2006, 2007) to Bayesian games and define two (complementary) equilibrium concepts, *Unacclimating Personal Equilibrium (PE/PPE) profile* and *Choice-Acclimating Personal Equilibrium (CPE) profile.* Although these concepts are already used in the literature, their formal definition and discussion are lacking.

Our model of Bayesian games might contribute to the existing literature in game and decision theory in at least two ways. First, it might improve on conventional game theory from a descriptive point of view. By applying Kőszegi and Rabin's framework of reference-dependent preferences, the notion of loss aversion is naturally transferred to a game-theoretic setting. By estimating appropriate consumption utility and gain-loss utility functions, this improvement is likely to have a positive impact on the predictive value of game theory.

Moreover, the distinction between PE and CPE profiles adds a new psychological dimension of decision making to solution concepts for Bayesian games. Second, our model allows to test the predictions of Kőszegi and Rabin in game theory, where a vast amount of experimental and empirical data might serve as valuable benchmarks and thus helps to assess the value of this theory, which has rapidly gained in importance over the past few years.

We reason that, in addition to the timing-feature highlighted by Kőszegi and Rabin, there are several other characteristics of decision situations that—at least—plausibly play a role in determining the appropriate solution concept, but have not been discussed in the literature so far. Clearly, a more thorough understanding of the underlying decision process is needed, and we hope to foster an interdisciplinary debate between psychologists and economists.

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Appendix A. Proofs

Proof of Proposition 1. The proof is analogous to the proof given in Milgrom and Weber (1985).

Note that because the functions μ and m are continuous we have that for each player $i \in I$ each type vector $\theta \in \Theta$ and any strategy profile $s \in S$ the functions $U_i^{\theta_i}(\cdot, \cdot)$ and $U_i^{\theta_i}(\cdot, s)$ are continuous. Since the action space A is finite Lusin's Theorem yields that the payoffs are equicontinuous. On the other hand, the finite type space Θ guarantees that the information structure is absolutely continuous. Then, the result follows from the Existence Theorem (Theorem 1) in Milgrom and Weber (1985). □

Proof of Proposition 2. The proof is completely analogous to the proof of Proposition 5 in Maskin and Riley (2000), replacing the von Neumann-Morgenstern utility by the appropriate Köszegi-Rabin utility. □

Proof of Proposition 3. The proof consists of two parts: First, we show that $\beta^{FA,PE}$ constitutes the unique symmetric PE profile, and second, we prove that given this profile, unilateral deviations cannot result in an asymmetric PE profile which implies that $\beta^{FA,PE}$ is even a unique symmetric PPE profile.

Given the bids of the other bidders, the first-order condition for i's bid b being a PE is that

$$\frac{\partial U(q|b)}{\partial q} = (v-q)h(q) - H(q) + \lambda v H(b)h(q) + \gamma v(1-H(b))h(q) = 0 \tag{A.1}$$

at $q = b$. This yields the necessary condition

$$b = v[1 + \lambda H(b) + \gamma(1 - H(b)] - \frac{H(b)}{h(b)} = v\big[1 + \gamma + (\lambda - \gamma)H(b)\big] - \frac{H(b)}{h(b)}. \tag{A.2}$$

Now assume that the other $n-1$ bidders bid according to a strictly increasing bidding function $\beta : [0, \overline{v}] \to \mathbb{R}_0^+$.[13] Let $G(v) = F^{n-1}(v)$ and $g(v) = (n-1)f(v)F^{n-2}(v)$ denote the distribution and density of the other bidders' highest intrinsic valuation. Thus, by the definition of the bidding function, we have $H(b) = G(\beta^{-1}(b))$ and $h(b) = g(\beta^{-1}(b))\frac{d\beta^{-1}(b)}{db}$.

[13] We will later show that the resulting symmetric equilibrium bidding function is indeed monotonic.

If bidder i also bids according to β, it is $H(b) = G(v)$ and $h(b) = \frac{g(v)}{\beta'(v)}$. Substituting these equalities in equation (A.2) yields the first order inhomogeneous differential equation

$$\beta(v) = v(1 + \lambda G(v)) - \frac{G(v)}{g(v)}\beta'(v) \tag{A.3}$$

which can be rewritten as

$$\beta'(v) + \frac{g(v)}{G(v)}\beta(v) = [1 + \lambda G(v) + \gamma(1 - G(v))]v\frac{g(v)}{G(v)} \tag{A.4}$$

By multiplication with $G(v)$ and by the product rule we obtain

$$\frac{d(G(v)\beta(v))}{dv} = [1 + \lambda G(v) + \gamma(1 - G(v))]vg(v). \tag{A.5}$$

Integrating from 0 to v and using the boundary condition $\beta(0) = 0$ leads to

$$\beta(v) = \frac{\int_0^v s\big[1 + \lambda G(s) + \gamma(1 - G(s))\big]g(s)ds}{G(v)} \tag{A.6}$$

$$= \frac{\int_0^v s\big[1 + \gamma + (\lambda - \gamma)G(s)\big]g(s)ds}{G(v)}. \tag{A.7}$$

That is, $\beta^{FA,PE}$ satisfies the necessary condition (A.2). Note that, since $vG^2(v) = (vG(v))G(v) > \int_0^v sG(s)g(s)ds$, we have

$$\beta'(v) = \frac{\big[v\left(1 + \gamma + (\lambda - \gamma)G(v)\right)G(v) - \int_0^v s\left(1 + \gamma + (\lambda - \gamma)G(s)\right)g(s)ds\big]\, g(v)}{G^2(v)} > 0,$$

that is, β is indeed strictly increasing. We rewrite β by dividing the integral into two parts

$$\beta(v) = \frac{(1 + \gamma)}{G(v)}\int_0^v sg(s)ds + \frac{(\lambda - \gamma)}{G(v)}\int_0^v sG(s)g(s)ds. \tag{A.8}$$

and apply partial integration to the second integral:

$$\int_0^v sG(s)g(s)ds = (\lambda - \gamma)G(v)vG^2(v) - \left(\int_0^v G^2(s)ds + \int_0^v sG(s)g(s)ds\right),$$

which yields

$$2\int_0^v sG(s)g(s)ds = (\lambda-\gamma)G(v)vG^2(v) - \int_0^v G^2(s)ds.$$

Inserting this and the equality $\frac{\int_0^v sg(s)ds}{G(v)} = \int_0^v 1-\frac{G(s)}{G(v)}ds$ yields an alternative form of β:

$$\begin{aligned}
\beta(v) &= (1+\gamma)\int_0^v 1-\frac{G(s)}{G(v)}ds + \frac{\lambda-\gamma}{2G(v)}\left(vG^2(v) - \int_0^v sG^2(s)ds\right) \\
&= v\left(1+\frac{\lambda}{2}G(v)+\frac{\gamma}{2}(1-G(v))\right) - \frac{\int_0^v s\left(1+\frac{\lambda}{2}G(s)+\frac{\gamma}{2}(1-G(s))\right)G(s)ds}{G(v)}.
\end{aligned}$$

The second-order condition for β being a PE is that

$$\frac{\partial^2 U(q|b)}{\partial q^2} < 0$$

at $q=b=\beta(v)$. Using the envelope theorem, we have at the point $q=b=\beta(v)$

$$\frac{d\left(\frac{U(q|b)}{\partial q}\right)}{dv} = \frac{\partial^2 U(q|b)}{\partial q\partial v} + \left(\frac{\partial^2 U(q|b)}{\partial q^2} + \frac{\partial^2 U(q|b)}{\partial q\partial b}\right)\beta'(v) = 0,$$

which implies

$$\frac{\partial^2 U(q|b)}{\partial q^2} = -\frac{\frac{\partial^2 U(q|b)}{\partial q\partial v}}{\beta'(v)} - \frac{\partial^2 U(q|b)}{\partial q\partial b} < 0,$$

since β is strictly increasing and both $\frac{\partial^2 U(q|b)}{\partial q\partial v} = h(b)(1+\lambda H(b))$ and $\frac{\partial^2 U(q|b)}{\partial q\partial b} = h(q)h(b)\lambda v$ are positive. This completes the first part of the proof, namely that $\beta^{FA,PE}$ constitutes a symmetric PE profile.

In the second step, we will show that if all $n-1$ other bidders bid according to the bidding function $\beta^{FA,PE}$, then there exists only one PE for bidder i, namely to bid $\beta^{FA,PE}(v)$. Assume that bidder i bids

$$\beta(x) := \frac{\int_0^x s[1+\gamma+(\lambda-\gamma)G(s)]g(s)ds}{G(x)}$$

with $x\in[0,\overline{v}]$ instead. That is, i bids according to the equilibrium function but does not necessarily reveal her true valuation v. Recall that β is continuous and increasing which implies $\beta([0,\overline{v}]) = [0,\beta(\overline{v})]$. That is, β covers the whole interval of possible bids. Thus, $\beta(x)$ can be used for every relevant bid, which can be interpreted as imitating the PE-bid of a bidder with valuation x.

A necessary condition for $\beta(x)$ constituting a PE is that it has to satisfy the first order condition (A.2), that is

$$\beta(x) = v\big[1 + \gamma + (\lambda - \gamma)H(\beta(x))\big] - \frac{H(\beta(x))}{h(\beta(x))}.$$

Since the other bidders choose the PE-bid according to $\beta^{FA,PE}$, it is $H(b) = G(\beta^{-1}(\beta(x))) = G(x)$ and thus we have the necessary condition

$$\beta(x) = v\big[1 + \gamma + (\lambda - \gamma)G(x)\big] - \frac{G(x)}{g(x)}. \tag{A.9}$$

Since in the first step, we have already shown that bidding $\beta(x)$ is a PE for a bidder with valuation $v = x$, equation (A.9) has to hold for $v = x$. But, since a variation in v only changes the first term on the right hand side of (A.9) and the right hand side is strictly increasing in v, equation (A.9) can only be fulfilled for $x = v$. Hence, if all other bidders choose the PE-bid $\beta^{FA,PE}(v_j)$, $j \neq i$, there exists a unique PE-bid for our considered bidder, namely to bid $\beta^{FA,PE}(v_i)$. Thus, $\beta^{FA,PE}$ also constitutes a PPE-profile and, since the symmetric PE profile is unique, also the unique symmetric PPE profile. □

Proof of Proposition 4. The proof is analogous to Lange and Ratan (2010), Proposition 1, by maximizing the utility

$$U(b|b) = (v - b)H(b) - \lambda v H(b)(1 - H(b)) + \gamma v(1 - H(b))H(b)$$

with respect to b. □

Proof of Corollary 1. Taking the difference

$$\beta^{FA,PE}(v) - \beta^{FA,CPE}(v) = \min\left\{\frac{\int_0^v s\big(\lambda - (\lambda - \gamma)G(s)\big)g(s)ds}{G(v)}, \beta^{FA,PE}(v)\right\}$$

immediately yields the result. □

Proof of Proposition 5. Fix any bidder $i \in \{1, \dots, n\}$. Given the bids of the other bidders, the first order condition for i's bid b being a PE is that

$$\begin{aligned}\frac{\partial U(q|b)}{\partial q} &= (v-q)h(q) + \lambda v H(b)h(q) + \gamma v(1-H(b))h(q) \\ &= h(q)\big(v[1+\lambda H(b) + \gamma(1-H(b))] - q\big) = 0\end{aligned}$$

is fulfilled at $q = b$ which yields the necessary condition

$$b = v\big(1 + \lambda H(b) + \gamma(1-H(b))\big) = v\big(1+\gamma+(\lambda-\gamma)H(b)\big). \tag{A.10}$$

That is, independent of the bids of the other bidders, i's optimal bid has to lie in the interval $[v(1+\gamma), v(1+\lambda)]$. Let $G(v) = F^{n-1}(v)$ and $g(v) = (n-1)f(v)F(v)$ denote the distribution and density of the other bidders' highest intrinsic valuation. Using the necessary condition (A.10), it is obvious that the only candidate for a symmetric PE profile in increasing bidding functions is

$$\beta^{SA,PE} = v\big(1 + \lambda G(v) + \gamma(1-G(v))\big) = v\big(1+\gamma+(\lambda-\gamma)G(v)\big) \tag{A.11}$$

with the derivative

$$\beta'(v) = 1 + \lambda G(v) + \gamma(1-G(v)) = 1+\gamma+(\lambda-\gamma)G(v) > 0. \tag{A.12}$$

Using the envelope theorem, we have

$$\frac{\partial^2 U(q|b)}{\partial q^2} = -\frac{\frac{\partial^2 U(q|b)}{\partial q \partial v}}{\beta'(v)} - \frac{\partial^2 U(q|b)}{\partial q \partial b} < 0,$$

since β is strictly increasing, $\frac{\partial^2 U(q|b)}{\partial q \partial v} = h(q)\big(1+\lambda H(b)+\gamma(1-H(b))\big) > 0$ and $\frac{\partial^2 U(q|b)}{\partial q \partial b} = h(q)h(b)v(\lambda-\gamma) > 0$. That is, $\beta^{SA,PE}$ satisfies also the second order condition, and thus, is the unique symmetric PE profile. It remains to show that $\beta^{SA,PE}$ also constitutes a PPE profile. Assume that all bidders but i bid according to $\beta^{SA,PE}$ and suppose that i deviates from her symmetric PE bid $\beta^{SA,PE}(v)$ to

$$\beta(x) = x\big(1+\lambda G(x)+\gamma(1-G(x))\big) = x\big(1+\gamma+(\lambda-\gamma)G(x)\big), \quad x \in [0, \overline{v}]$$

by imitating a bidder with valuation x. Note that $\beta([0,\overline{v}]) = [0,\overline{v}(1+\lambda)]$. Thus, we can use $\beta(x)$ for every relevant bid. If there exists another $b = \beta(x)$ that constitutes a PE for bidder i, then it has to fulfil the necessary condition (A.10), that is $b = v\big(1+\gamma+(\lambda-\gamma)H(b)\big)$. But since the other bidders bid according to $\beta^{SA,PE}$, it is $H(b) = G(x)$ and, thus,

$$\beta(x) = v\big(1+\lambda G(x)+\gamma(1-G(x))\big) = v\big(1+\gamma+(\lambda-\gamma)G(x)\big), \quad x \in [0,\overline{v}]$$

However, $\beta(x)$ is the PE bid of a bidder with valuation x which implies $x\big(1+\gamma+(\lambda-\gamma)G(x)\big) = v\big(1+\gamma+(\lambda-\gamma)G(x)\big)$, and thus $x = v$ Hence, if all other bidders choose the PE-bid $\beta^{SA,PE}(v_j)$, $j \neq i$, there exists a unique PE-bid for our considered bidder, namely to bid $\beta^{SA,PE}(v_i)$. Thus, $\beta^{SA,PE}$ also constitutes a symmetric PPE-profile and thus the unique PPE profile. □

Proof of Proposition 6. The proof is analogous to Lange and Ratan (2010), Proposition 3, by maximizing the utility

$$U(b|b) = \int_0^b (v-p)h(p)dp - \lambda v H(b)(1-H(b)) + \gamma v(1-H(b))G(b)$$

with respect to b. □

Proof of Corollary 2. Taking the difference

$$\beta^{SA,PE}(v) - \beta^{SA,CPE}(v) = \min\big\{v\big(\gamma+(\lambda-\gamma)(1-G(v))\big),\ \beta^{SA,PE}(v)\big\}$$

immediately yields the result. □

Appendix B. Additional analysis

We now consider a second-price auction with induced values. That is, for each bidder the auctioned good has a fixed private monetary value and thus there is only one dimension of consumption.

In Proposition 7 we state that with induced values the bidders have no incentive to overbid and that truthful bidding is the unique PE. This confirms the result in Lange and Ratan

(2010) which states that with induced values incentive compatible bidding constitutes the unique CPE (at least for bidders with a valuation above some threshold).

Proposition 7. *In the second-price auction, the bidding function $\beta(v) = v$ constitutes the unique symmetric PE profile.*

Proof. Within the induced values framework, bidding truthfully first-order stochastically dominates any other bid (in the weak sense). Since the PE concept does not allow the choice of first-order dominated lotteries, $\beta(v) = v$ constitutes the unique symmetric PE profile. In the following we give a detailed proof.

Fix any bidder $i \in \{1, \dots, n\}$ with private signal v. Bidder i's expected KR-utility $U(q|b)$ from bidding q while her reference point is determined by the bid b is given for the six cases (1) $v \geq b \geq q$, (2) $v \geq q > b$ (3) $q < v \leq b$, (4) $b < v \leq q$, (5) $v \leq b < q$ and (6) $v \leq q < b$ as follows.

(1) and (2)

$$\begin{aligned} U(q|b) \quad = \quad & \int_0^q \left((v-p) + \gamma \int_p^{\max\{b,p\}} (s-p)h(s)ds - \lambda \int_0^p (p-s)h(s)ds \right) h(p)dp \\ & -\lambda(1-H(q)) \int_0^b (v-p)h(p)dp + \gamma(1-H(b)) \int_0^q (v-p)h(p)dp \end{aligned}$$

(3)

$$\begin{aligned} U(q|b) \quad = \quad & \int_0^q \left((v-p) + \gamma \int_p^{\max\{b,p\}} (s-p)h(s)ds - \lambda \int_0^p (p-s)h(s)ds \right) h(p)dp \\ & -(1-H(q)) \left(\lambda \int_0^b (v-p)h(p)dp - \gamma \int_v^b (p-v)h(p)dp \right) \\ & +\gamma(1-H(b)) \int_0^q (v-p)h(p)dp \end{aligned}$$

(4)

$$\begin{aligned} U(q|b) \quad = \quad & \int_0^q \left((v-p) + \gamma \int_p^{\max\{b,p\}} (s-p)h(s)ds - \lambda \int_0^p (p-s)h(s)ds \right) h(p)dp \\ & -(1-H(q)) \left(\lambda \int_0^v (v-p)h(p)dp \right) \\ & +(1-H(b)) \left(\gamma \int_0^v (v-p)h(p)dp - \lambda \int_v^q (p-v)h(p)dp \right) \end{aligned}$$

(5) and (6)

$$\begin{aligned}U(q|b) &= \int_0^q \left((v-p) + \gamma \int_p^{\max\{b,p\}} (s-p)h(s)ds - \lambda \int_0^p (p-s)h(s)ds \right) h(p)dp \\ &\quad -(1-H(q)) \left(\lambda \int_0^b (v-p)h(p)dp - \gamma \int_v^b (p-v)h(p)dp \right) \\ &\quad +(1-H(b)) \left(\gamma \int_0^v (v-p)h(p)dp - \lambda \int_v^q (p-v)h(p)dp \right)\end{aligned}$$

For the partial derivatives in the above cases with repect to q we have

(1)

$$\frac{\partial U(q|b)}{\partial q} = h(q) \left((v-q)\big(1+\gamma(1-H(b)) + \lambda H(q)\big) + \lambda \int_q^b (v-s)h(s)ds \right)$$

(2)

$$\frac{\partial U(q|b)}{\partial q} = h(q) \left((v-q)\big(1+\gamma(1-H(b)) + \lambda H(b)\big) + \lambda \int_b^q (q-s)h(s)ds \right)$$

(3)

$$\frac{\partial U(q|b)}{\partial q} = h(q) \left((v-q)\big(1+\gamma(1-H(v)) + \lambda H(q)\big) + \lambda \int_q^b (v-s)h(s)ds \right)$$

(4)

$$\frac{\partial U(q|b)}{\partial q} = h(q) \left((v-q)\big(1+\gamma(1-H(v)) + \lambda H(b)\big) + \lambda \int_b^q (q-s)h(s)ds \right)$$

(5)

$$\frac{\partial U(q|b)}{\partial q} = h(q) \left((v-q)\big(1+\lambda\big) - \lambda \int_b^q (q-s)h(s)ds - \gamma \int_v^b (s-v)h(s)ds \right)$$

(6)

$$\frac{\partial U(q|b)}{\partial q} = h(q) \left((v-q)\big(1+\lambda\big) - \lambda \int_q^b (v-s)h(s)ds - \gamma \int_v^q (s-v)h(s)ds \right)$$

The first order condition for a symmetric PE is that $\frac{\partial U(q|b)}{\partial q} = 0$ at $q = b$. It can be easily seen that this condition hast only the solution $q = b = v$. Thus, only the bidding function $\beta(v) = v$ satisfies the first order condition.

A sufficient condition for $\beta(v) = v$ being a PE is that the partial derivative $\left.\frac{\partial U(q|b)}{\partial q}\right|_{b=v}$ at $b = v$ is positive for $q < v$ and negative for $q > v$. This holds, since the derivatives in (1) and (3) are positive and in (5) negative. $\square$