Reference-Dependent Bidding in Dynamic Auctions[☆]

Karl-Martin Ehrhart[a], Marion Ott[b]

[a] *Karlsruhe Institute of Technology (KIT), Germany*
[b] *RWTH Aachen University, Germany*

Abstract

Loss-averse bidders face different sensations as the price clock proceeds in single-unit ascending or descending auctions. We investigate equilibrium bidding behavior of bidders with independent private values and reference-dependent preferences, applying the Kőszegi and Rabin (2006) model. Bidders' stochastic reference points are endogenous, and are determined by their strategy and their beliefs about the other bidders. Utility functions reflect that bidders anticipate changes in their reference point due to updated beliefs, e.g. about the own winning probability, during the course of the auction. An optimal bidding strategy can be reduced to a series of optimal binary decisions at each price, i.e., approve or quit in the English Auction (EA) and wait or bid in the Dutch Auction (DA). We solve for personal equilibrium (PE) profiles, which contain for each bidder a bidding strategy that is optimal given the others' bidding strategies and the reference point induced by the own and others' strategies. There exists a range of belief-free PE profiles in the EA and a range of symmetric PE profiles in the DA under different existence conditions. If symmetric PE profiles exist in both auctions, the expected revenue in the DA is higher than in the EA. The difference is mainly driven by the aversion to losing the item in the DA.

Keywords: Reference-dependent preferences, endogenous reference point, English auction, Dutch auction, loss aversion

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Email addresses: `ehrhart@kit.edu` (Karl-Martin Ehrhart), `marion.ott@rwth-aachen.de` (Marion Ott)

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1. Introduction

Bidders who participate in an English auction or in a Dutch auction face different auction processes that go along with quite different sensations. In an English auction, a bidder who desires the auctioned good asks himself at any announced auction price whether he prefers to approve the price or whether he is ready to quit. When he quits, he might be disappointed that he did not get the good, but in quitting he accepts not getting the item. In the Dutch auction, however, this bidder knows at any auction price at which he does not bid that another bidder might snatch the good away. Thus, the mechanics behind sensations of losing the good are different from auction to auction. Similarly, sensations of gaining the good or of gaining or losing money are evoked in different ways in these auctions. Loss-averse bidders might adjust bids to such sensations and these adjustments might follow quite different rules in the auctions. English auctions and Dutch auctions generate the same expected revenue with risk-neutral symmetric bidders when bidders have independent private values. How does this change when bidders are loss-averse? How do different processes affect a loss-averse bidder and the expected revenue?

We analyze the single-unit English clock auction (EA) and the Dutch auction (DA). Bidders have independent private values and they are risk-neutral. The bidders have reference-dependent preferences and their reference point is endogenously determined by their planned bid. The basic preference model that we apply is the endogenous reference point model of Kőszegi and Rabin (2006, 2007) and their criterion (unacclimating) personal equilibrium for optimal decision making. A bid has to be optimal given the reference-point, and the reference-point has to be induced by this bid. A planned bid induces beliefs about winning the auction, which in turn induce expectations about gains or losses. Bidders may be loss-averse with respect to the auctioned good and money.

The methodological innovations of our analysis stem from three features of the model. First, we adapt the endogenous reference-point formulation of Kőszegi and Rabin (2007) to our dynamic mechanisms. The bidders are Bayesians that use all the probabilistic information they have correctly. Given their planned bid and their beliefs about the other bidders, they perfectly anticipate their future updating of these beliefs.[1] When the auction proceeds, the beliefs about

[1]Due to the differences in settings, our approach differs from another dynamic approach with loss-averse decision makers by Kőszegi and Rabin (2009), who analyze optimal decisions on multi-period consumption plans by a

winning the auction will change. The probability of winning with a certain bid decreases during the course of an English auction and increases during the course of a Dutch auction. This in turn impacts the gain-loss utility of loss-averse bidders, and our bidders condition their sensation about potential gains and losses on the upcoming bidding stages being reached. This updating is reflected in their utility functions.

Second, the analysis takes the auction process into account and incorporates a simple but plausible decision process, which is ignored (and can be ignored) in the usual analysis with gain-loss-neutral bidders. We call this process *binary bidding*. At each price in the auction, the bidder faces a binary decision (to quit or approve in the EA and to wait or to bid in the DA) and has to choose the better option. Rather than imposing less cognition by focusing on binary decisions, we show that the optimal decision of a Bayesian bidder about a planned bid that he will follow through can be reduced to a sequence of such binary decisions given that his planned bid induces his reference point. That is because the bidder's utility function incorporates his correct anticipation about the future, given the current information. For gain-loss-neutral bidders, the binary decisions are in the EA obviously in line with their optimal bidding limit and in the DA the information that can be gained does not impact his expected utility.

Third, we solve for equilibria for a dynamic game and therefore need to adjust or introduce definitions for optimal individual decisions (personal equilibrium) and mutual best responses (personal equilibrium profiles).

In an EA, a bidder can at every stage decide whether to approve the auction price or to quit the auction. While he cannot control at what price he wins, he can perfectly control at what price he gives up any chance of winning the auction and gives up the good. In this auction, losses in the money dimension and gains in the good dimension will play a major role. In a DA, the bidder decides at any auction price whether to wait or to bid. He can control at what price he wins the auction but he cannot control at what price he loses the auction and the good. In this auction, gains in money and losses in the goods dimension are predominant.

A bid is a PE if it is optimal given the reference point that it induces and given beliefs about the others' bids. At any earlier auction price, the bidder prefers to approve in the EA or to wait

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decision maker that faces changes in wealth.

in the DA. If the auction price actually reaches the bid, the bidder is indifferent between his two options, but any further approving or waiting would make him worse off. In both auctions, for given beliefs about the others' bids, loss-averse bidders usually have more than one PE. That is, a bidder who has a bid in mind and is willing to follow this bid through would, for the same beliefs about others, also be willing to follow another bid through if he had that other bid in mind (as a reference bid). We find that a bidder's PE form a continuous interval. In the EA, the PE and the interval of PE depend on a bidder's valuation and his gain-loss parameters. In the DA, PE additionally depend on the beliefs about the opponents bids. PE in the EA and the DA exist for different but overlapping gain-loss parameter regions.

Deriving PE profiles, which require that beliefs are consistent with the opponents' bidding functions, we also find intervals. In the EA, any profile of PE is a PE profile. In the DA, we need to assume symmetric bidders and find symmetric PE profiles, which again form an interval.

A bidder who takes the additional step to compare his PE may find it hard to selecting from his PE. The PE that provides the highest expected utility before the auction starts will usually not be the best throughout the auction, and the bidder anticipates this. While at the beginning bidding late may seem attractive, once some time has passed bidding earlier may be better. Also, once a bidder has reached his bid, a later bid may on the way to its final decision look better for some time but then provide negative utility for some period. Thus, if there is no PE that provides the highest utility until it is reached and that does not evoke any regret once it is reached, then any comparison must neglect some relevant features of bids. Only in the EA for a bidder who is gain-loss-neutral with respect to money we find such a best PE, called preferred personal equilibrium (compare Kőszegi and Rabin, 2007, for the term).

If PE in both auctions exist, the expected revenue in the highest symmetric PE profile in the EA is not higher than the expected revenue in the lowest symmetric PE in the DA.

Related papers that analyze auctions when bidders have reference-dependent preferences and when reference points are endogenous are by Lange and Ratan (2010), Eisenhuth (2010), and Belica and Ehrhart (2013). They all focus on static auctions and the first two papers also apply the choice-acclimating equilibrium (Kőszegi and Rabin, 2007) that we consider less appropriate for our setting. Reference-dependent preferences with endogenous reference points are applied to other games for example by Herweg et al. (200) and Rosato (2014). Investigations on equilibria in games when players have endogenous reference points are provided by Shalev (2000) and Belica and Ehrhart (2013). Rosenkranz and Schmitz (2007) and Shunda (2009) analyze reference-dependent bidders in auctions when reference points are exogenously given.

in Section 2 we present the model, including the utility functions and definitions of solution concepts. We analyze the English and the Dutch auction sequentially in Section 3. For both auctions, we first analyze individual decisions taking the distribution of others' bids (the winning probability) as given, and then proceed to interactive decision making, that is, to equilibria of the auction games. A comparison of auction revenues is given in Section 4.4. Section 5 concludes.

2. The Model

2.1. Reference-Dependent Utility

We build upon the gain-loss utility model of Kőszegi and Rabin (2006). Decision makers evaluate gains and losses in different consumption spaces separately. In our model, we have two consumption spaces, the good g and money m. A decision maker's expected utility is given by

$$U(X_c|X_r) = \int \int u(c|r) dR(r) dC(c), \tag{1}$$

where the random variables X_c and X_r with distribution functions C and R represent the stochastic consumption and reference outcomes, respectively. The utility $u(c|r) = u((c_g, c_m)|(r_g, r_m))$ from consumption levels c_g of the good and c_m of money, when the respective reference levels are r_g and r_m is

$$u(c|r) = \sum_{k \in g,m} w_k(c_k) + \mu_k(w_k(c_k) - w_k(r_k)).$$

We assume risk-neutral, gain-loss averse decision makers with *consumption utility functions* $w_k(c_k)$ and *gain-loss utility function* $\mu_k(w_k(c_k) - w_k(r_k))$ for $k \in \{g, m\}$ of the following form:

$$\begin{aligned} w_m(c_g) &= c_g\, v \\ w_k(c_m) &= c_m \end{aligned} \qquad \text{and} \qquad \mu_k(x) = \begin{cases} \gamma_k x & \text{if } x > 0 \\ \lambda_k x & \text{if } x \leq 0 \end{cases} \quad \text{with } \lambda_k \geq \gamma_k \geq 0$$

for the *gain parameter* λ_k and the *loss parameter* γ_k.[2] The consumption utility function $w_k(c_k)$ gives the utility from enjoying consumption level c_k. In the auction, c_g can take the levels 0 if the bidder does not win, or 1 if he wins, and v is the decision maker's consumption value for the good. The consumption level c_m is his payment for the good, that is, $c_m = -p$ if he pays p.[3] The gain-loss utility $\mu_k(w_k(c_k) - w_k(r_k))$ is added to the consumption utility. It increases the utility $u(c|r)$ if the consumption level is above the reference level, $w_k(c_k) > w_k(r_k)$, such that the decision maker experiences a gain relative to the reference point, but it decreases $u(c|r)$ if the consumption level is below the reference level, $w_k(c_k) < w_k(r_k)$, such that he experiences a loss relative to the reference point. The assumptions on the parameters λ_k and γ_k ensure that a loss of size x looms larger than a gain of the same size x.

We will give specific forms of the utility function $U(\cdot|\cdot)$ for our auction setting after having introduced the game.

2.2. The Bidders, Auctions, Bids, and Reference Points

In the auction game n bidders participate, a bidder's strategy is his bid b or his bidding function $\beta(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$, and his utility is represented by the utility function $U(\cdot|\cdot)$. We analyze an independent private values model. Each bidder's value v for the good is independently drawn from a distribution F with density f and full support on $[0, \bar{v})$. We call the bidders *symmetric*, if their values are drawn from the same distribution F and if their parameters λ_k and γ_k, $k \in \{g, m\}$ are the same.[4]

The auctions are dynamic clock auctions with a continuous price clock. In the English Auction (EA), a bidder faces a sequence of binary choice problems: At every auction price p, the bidder has to decide either to accept p and stay in the auction – to approve – or to drop out of the auction – to quit. Once a bidder has quit the auction he may not return to the auction. The auction

[2] Note the difference to the notation used in most of the related papers (e.g. Kőszegi and Rabin, 2006), where $\mu(x) = \eta x$ for $x > 0$ and $\mu(x) = \eta\lambda x$ for $x \leq 0$ with $\eta > 0$ and $\lambda > 1$, such that η corresponds to our gain parameter γ_k for all k and $\eta\lambda$ corresponds to our loss parameter λ_k for all k. Our notation is closer to that of Lange and Ratan (2010) who assume $\gamma_k = 0$.

[3] Money is the numeraire and bidders are risk-neutral in money. Bidders do not possess any unit of the item before the auction, i.e. their initial endowment of the item is zero (compare, e.g. Lange and Ratan, 2010). For risk-neutral bidders, adding positive endowments does not make a difference in our analyses.

[4] We will assume symmetric bidders in the analysis of the DA and for comparisons between auctions.

stops when one of two remaining bidders quits. The other bidder then becomes the winner of the auction and has to pay the price at which the auction stopped.

The decision to approve p is part of a plan to stay in the auction until the auction price reaches a upper bidding limit $b > p$. We call this plan to approve all prices up to b in the EA *the bid* b. The bid b induces a random variable $A(p, b)$, which depends on the current price p and which maps the stochastic outcome from the bid b to the consumption utilities from winning the good at prices between p and b and not winning the good, if the price exceeds b. $A(p, p)$ represents the consumption utility zero from the certain outcome not to win the auction when quitting at p when the auction has reached p.

When deciding about approving or quitting, the bidder takes his stochastic reference point R into account. R is a random variable that represents the consequences of a decision. A decision is a bid b, and the distribution over possible consumption utilities, which changes in p, is $A(p, b)$. Therefore, $R = A(p, b)$.

We denote by $U(A(p, b)|A(p, \hat{b}))$ the loss-averse bidder's utility from the bid b, when the EA reaches the price p and his reference point is induced by the plan to bid $\hat{b}$.

In the Dutch Auction (DA), like in the EA, the bidders face binary choice problems at every price p. However, in contrast to the EA, the auction price p continuously decreases. At every price p, each bidder has to decide to bid or to wait. Once a bidder bids, he wins the auction, the auction stops, and he has to pay the price at which the auction stops. Waiting at p is part of a plan to wait until a price $b < p$, which we call *the bid* b in the DA. The bid b induces a random variable $W(p, b)$, which depends on the current price p and which which maps the stochastic outcome from the bid b to the consumption utilities from winning the good at the price b or not winning the good, if another bidder bids between p and b. $W(p, p)$ represents the certain consumption utility $v - p$ from the certain outcome of winning the auction with the bid p when the auction has reached p. A stochastic reference point R in the DA equals $W(p, b)$.

We denote by $U(W(p, b)|W(p, \hat{b}))$ the loss-averse bidder's utility from the bid b, when the DA reaches the price p and his reference point is induced by the plan to bid $\hat{b}$.

2.3. Beliefs

For our analysis, a bidder's beliefs about the bid of his strongest competitor are relevant. We introduce different notation for the EA and the DA, because updating of the beliefs differs in the two auctions due to the inverse price dynamics.

In the EA, let $H(b)$ denote the belief that the strongest opponent bids less than b, that is, the subjective probability of winning the auction with the bid b, before the auction starts. Denote the corresponding density by $h(\cdot)$. The auction starts at $p = 0$, where $H(0) = 0$. $H(b|p)$ denotes the conditional subjective probability of winning with the bid b when the auction is at price level p. Thus, $H(b|p) = 0$ for all $b \leq p$. We assume that a bidder cannot observe other bidders' dropping out of the auction.[5] Hence, the conditional probability is calculated as

$$H(b|p) = \frac{H(b) - H(p)}{1 - H(p)}, \tag{2}$$

which implies $H(p|p) = 0$. This updating from $H(\cdot) = H(\cdot|0)$ to $H(\cdot|p)$ truncates the probability function $H(\cdot)$ at p from below and shifts the probability mass to prices larger than p. We assume that H with density h is a continuous distribution with full support on $[0, y]$ for $y > 0$.[6] The conditional density is

$$\frac{\partial H(b|p)}{\partial b} = h(b|p) = \frac{h(b)}{1 - H(p)} > 0$$

and it holds that

$$\frac{\partial H(b|p)}{\partial p} = \frac{-h(p)(1 - H(p)) - (H(b) - H(p))(-h(p))}{(1 - H(p))^2} = -\frac{h(p|p)(1 - H(b))}{1 - H(p)} < 0.$$

In the DA, before the auction starts, a bidder's belief that the highest bid of the others is below b is given by $G(b)$, the subjective probability of winning the auction with the bid b. Denote

[5] If, to the contrary, bidders could observe others' quitting, then the updating would need to take this information into account. $H(b|p)$ would then not account for the fact that at least one more bidder is still in the auction and will bid more than p but it would account for the true number of bidders that is still in the auction and that, thus, will bid more than p. In our private values model a bidder knows his value and thus cannot infer information about the value from the others bids. However, his belief about winning is different, if for example he observes all but one opponent quitting very early compared to no observation of quitting. Because our analysis shows that bids will be independent of beliefs (but dependent on the own parameter values and the own value for the good), the results would not change if bidders could observe when opponents quit.

[6] For example, with symmetric bidders and symmetric monotonic bidding functions this holds true.

the corresponding density by $g(\cdot)$. When the auction procedure starts at the sufficiently high price $\bar{p}$ we have $G(\bar{p}) = 1$. The conditional probability $G(b|p)$ of winning with the bid b when the auction is at price level $p \geq b$ is

$$G(b|p) = \frac{G(b)}{G(p)}, \tag{3}$$

which implies $G(p|p) = 1$. This updating from $G(\cdot) = G(\cdot|\bar{p})$ to $G(\cdot|p)$ truncates the probability function $G(\cdot)$ at p from above and shifts the probability mass to prices smaller than p. As we did for H, we assume that G, with density g is a continuous distribution with full support on $[0, y]$ with $y > 0$. The conditional density is

$$\frac{\partial G(b|p)}{\partial b} = g(b|p) = \frac{g(b)}{G(p)} > 0$$

and it holds that

$$\frac{\partial G(b|p)}{\partial p} = -\frac{G(b)g(p)}{G(p)^2} = -G(b|p)g(p|p) < 0.$$

2.4. Binary Bidding

In our dynamic auctions, the bidder has to decide at each price p between two options: approve or quit in the EA and wait or bid in the DA. This implies a process of continuously making binary decisions, which we call "binary bidding." A bid b in the EA therefore involves a decision to approve all prices $p \leq b$ and to quit at $p = b$. A bid b in the DA involves the decision to wait at all prices $p \geq b$ and to bid at $p = b$. Therefore, a utility maximizing bid b in the EA involves the preferability of approving over quitting at all prices $p \leq b$ and the preferability of quitting over approving at all prices $p > b$. A bid b in the DA involves the preferability of waiting over bidding at all prices $p \geq b$ and the preferability of bidding over waiting at all prices $p < b$.

The dynamics and the binary bidding process have consequences for the utility functions $U(\cdot|\cdot)$. For example, in the EA, a bidder's utility from a bid b when his reference bid is $\hat{b} > b$, is composed of his utility at any q along the price path between p and $\hat{b}$, weighted by the marginal subjective probabilities seen from p. That is, our utility functions are calculated by integrating over utilities of the form given in (1).

2.5. Utility Functions

The utility functions reflect the bidders' anticipation of the changing expected gains and losses relative to the reference point when the auction price develops.[7] Note that gaining the good relative to the reference point always comes with a loss of money, and losing the good relative to the reference point always comes with a gain of money. We derive the functions $U(A(p,b)|A(p,\hat{b}))$ for the EA and $U(W(p,b)|W(p,\hat{b}))$ for the DA in Appendix A.[8] They represent the preferences of a gain-loss averse bidder at price level p in the auction, who anticipates the further course of the auction and who evaluates for any $q > p$ in the EA ($q < p$ in the DA) the sensations of gains and losses that might realize at q, weighted by the marginal probability $h(q|p)$ in the EA ($g(q|p)$ in the DA) with that they become relevant. Consequently, the gains and losses at q are evaluated relative to the potential outcomes of the reference bid $\hat{b}$ seen from q.

The utility function $U(A(p,b)|A(p,\hat{b}))$ describes a bidder's expected utility from the bid b, when he has the reference bid $\hat{b}$ in mind and when the auction has reached the price level p in the EA. We get the following expressions for $p \leq \min\{b,\hat{b}\}$:[9]

$$U(A(p,b)|A(p,\hat{b})) = \begin{cases} \int_p^b v\left(1+\gamma_g(1-H(\hat{b}|s))\right) - s\left(1+\lambda_m(1-H(\hat{b}|s))\right) \mathrm{d}H(s|p) & \\ \quad + \int_b^{\hat{b}} \gamma_m s - \lambda_g v \,\mathrm{d}H(s|p) & \text{if } b \leq \hat{b} \\ \int_p^{\hat{b}} v\left(1+\gamma_g(1-H(\hat{b}|s))\right) - s\left(1+\lambda_m(1-H(\hat{b}|s))\right) \mathrm{d}H(s|p) & \\ \quad + \int_{\hat{b}}^b v(1+\gamma_g) - s(1+\lambda_m) \,\mathrm{d}H(s|p) & \text{if } b > \hat{b}. \end{cases} \tag{4}$$

[7] In particular, a bidder in the EA at p knows that if $q > p$ is reached, his probability of losing with the reference bid $\hat{b}$ will have increased from $1 - H(\hat{b}|p)$ to $1 - H(\hat{b}|q)$, thus, there will be stronger sensations of gaining the good if it is won at q than at earlier stages. A bidder in the DA at p knows that if $q < p$ is reached, his probability of winning with the reference bid $\hat{b}$ will have increased from $G(\hat{b}|p)$ to $G(\hat{b}|q)$, thus, there will be stronger sensations of losing the good if an opponent bids at q than at earlier stages.

[8] In Appendix B we additionally show how to derive the same functions $U(A(p,b)|A(p,\hat{b}))$ and $U(W(p,b)|W(p,\hat{b}))$ for a continuously increasing or decreasing price from an approach with discrete price steps.

[9] More generally, for all p:

$$U(A(p,b)|A(p,\hat{b})) = \begin{cases} \int_p^{\max\{p,b\}} v\left(1+\gamma_g(1-H(\hat{b}|s))\right) - s\left(1+\lambda_m(1-H(\hat{b}|s))\right) \mathrm{d}H(s|p) & \\ \quad + \int_{\max\{p,b\}}^{\max\{p,\hat{b}\}} \gamma_m s - \lambda_g v \,\mathrm{d}H(s|p) & \text{if } b \leq \hat{b} \\ \int_p^{\max\{p,\hat{b}\}} v\left(1+\gamma_g(1-H(\hat{b}|s))\right) - s\left(1+\lambda_m(1-H(\hat{b}|s))\right) \mathrm{d}H(s|p) & \\ \quad + \int_{\max\{p,\hat{b}\}}^{\max\{p,b\}} v(1+\gamma_g) - s(1+\lambda_m) \,\mathrm{d}H(s|p) & \text{if } b > \hat{b}. \end{cases}$$

Four important special cases for $p < b$ can be derived from (4):

$$U(A(p,p)|A(p,b)) = -\lambda_g v H(b|p) + \gamma_m \int_p^b s \,\mathrm{d}H(s|p) \tag{5}$$

$$U(A(p,b)|A(p,b)) = \int_p^b v\big(1+\gamma_g\,(1-H(b|s))\big) - s\big(1+\lambda_m\,(1-H(b|s))\big)\,\mathrm{d}H(s|p) \tag{6}$$

$$U(A(p,p)|A(p,p)) = 0 \tag{7}$$

$$U(A(p,b)|A(p,p)) = v(1+\gamma_g)H(b|p) - \int_p^b s(1+\lambda_m)\,\mathrm{d}H(s|p) \tag{8}$$

Equation (5) gives the utility from quitting now, at p, when the reference bid b is higher. The decision provides zero consumption utility, but a sensation of losing the good with the winning probability of b and the corresponding sensation of gaining money amounts q, $p < q \leq b$ with marginal probabilities $h(q|p)$. Quitting at p makes a sensation of winning the good impossible. If, as in (6), bid and reference bid are identical and above p, the decision involves no sensation of losing the good (and gaining money) because the auction is lost for sure if it does not end before b. The bidder has the consumption utility $v - q$ weighted by the marginal probability $h(q|p)$ of winning at any q, $p \leq q \leq b$. Also, at any such q and with the same weight, he has a sensation of losing the payments and gaining the good relative to the reference point with the reference point's probability $1 - H(b|q)$ of not winning the auction. Equation (7) provides the utility from quitting now, when the reference is to quit now. Both the bid and the reference bid generate no consumption utility for sure, and no sensation of gains or losses. In contrast, in (8), the bid b is evaluated relative to a reference point of quitting now, at p. This generates consumption utility $v - q$ as well as a sensation of gaining the good and losing the payment weighted by the marginal probability $h(q|p)$ of winning at any q, $p \leq q \leq b$. Relative to the reference bid, which involves not winning for sure, there are no sensations of gaining money or losing the good.[10]

The utility function $U(W(p,b)|W(p,\hat{b}))$ describes a bidder's expected utility from the bid b,

[10] One obvious difference to the second-price auction is that in our utility function (6) one additive term of a money-loss is completely missing as compared to the second-price auction (see Lange and Ratan, 2010). In the second-price auction, winning the auction at p comes with a sensation of losing money because the bidder assigns positive probability to winning at lower prices. In contrast, a bidder in the EA knows that if he will win at p, the auction will have reached the auction price p, and that he then will assign zero probability to winning at lower prices. The price comparisons that generate the term in the second-price auction would be backward looking in the EA and do not occur.

when he has the reference bid $\hat{b}$ in mind and when the auction has reached the price level p in the DA.[11]

$$U(W(p,b)|W(p,\hat{b})) = \begin{cases} (v-b)\,G(b|p) + (\lambda_g v - \gamma_m \hat{b})\,G(\hat{b}|p)\ln(G(b|p)) & \\ \quad + \gamma_g v\,[G(b|p) - G(\hat{b}|p)] - \lambda_m\,[b\,G(b|p) - \hat{b}\,G(\hat{b}|p)] & \text{if } b \geq \hat{b} \\ (v-b)\,G(b|p) - \lambda_g v\,G(\hat{b}|p)\,[1 - G(b|\hat{b}) - \ln(G(\hat{b}|p))] & \\ \quad + \gamma_m\,G(\hat{b}|p)\,[\hat{b}(1-\ln(G(\hat{b}|p))) - bG(b|\hat{b})] & \text{if } b < \hat{b}. \end{cases} \tag{9}$$

Expression (9) involves four special cases for $p > b$:

$$U(W(p,p)|W(p,b)) = v - p + \gamma_g v(1 - G(b|p)) - \lambda_m(p - bG(b|p)) \tag{10}$$

$$U(W(p,b)|W(p,b)) = (v-b)G(b|p) + (\lambda_g v - \gamma_m b)G(b|p)\ln(G(b|p)) \tag{11}$$

$$U(W(p,p)|W(p,p)) = v - p \tag{12}$$

$$U(W(p,b)|W(p,p)) = (v-b)G(b|p) - \lambda_g v(1 - G(b|p)) + \gamma_m(p - bG(b|p)) \tag{13}$$

Of these four cases only in (10) the reference bid is lower than the assessed bid, such that sensations of gaining the good and losing money are possible. The auction is won for sure, while with the reference it is lost with probability $1 - G(b|p)$, generating the sensation of gains and losses. In (12) the auction is won at price p for sure and in line with the reference bid. In (11) and (13) the auction is won with probability $G(b|p)$ with the bid b. Because the reference bids in (11) and (13) are weakly higher than b, sensations of gaining the good do not occur. In (11), bid and reference bid are equal and in the future. With probability $G(b|p)$ the bidder will win according to his reference bid but with anticipated updated probability $\ln(G(b|p))$ an opponent bids earlier, generating a sensation of losing the good. In (13) the reference bid wins for sure but the assessed bid wins only with probability $G(b|p)$, and, thus, does not win with probability $1 - G(b|p)$. The reference bid results in $v - p$ for sure while the assessed bid results in $v - b$ with probability $G(b|p)$, such that the difference generates a sensation of losing the good and gaining money.

[11] For $p < \max\{b, \hat{b}\}$, note that $G(b|p) = 1$ for $b < p$ and remember that $G(\hat{b}|p)G(b|\hat{b}) = G(b|p)$.

2.6. Optimal Choices: Personal Equilibria and Preferred Personal Equilibria

The preferability of a choice depends on the reference point, which in our auctions is determined by an own reference bid and beliefs about the highest bid of the others. A choice is considered optimal, if, given the reference point, all other options provide weakly lower utility. An optimal choice constitutes a Personal Equilibrium (PE), if the reference bid equals the chosen bid. This idea of endogenizing the reference point was introduced by Kőszegi and Rabin (2006) and Kőszegi and Rabin (2007). There may be several PE, each a best choice given the reference point that it induces. Because these PE may differ in the expected utility that they provide, Kőszegi and Rabin (2006) propose to refine to Preferred Personal Equilibria (PPE): decision makers choose a PE that provides the highest utility. This requires an additional level of rationality. Choice need not only be optimal given a reference point but they also need to maximize utility over all such optimal choices. During the course of an auction, different PE may maximize the utility, thus, it is not obvious how to apply the idea of a PPE. We define a PPE in a strict sense as a PE that maximizes expected utility throughout the auction and that may not cause regret when a decision became irreversible (having quit in the EA or having bid in the DA).

An alternative approach on individual choices when reference points are endogenous by Kőszegi and Rabin (2007) is the choice-acclimating equilibrium (CPE). This concept does not require that a choice is optimal given the reference point. It chooses the decision that maximizes utility over pairs of choice and reference point when choice and reference point are equal. Thus, in this concept, choice and reference point are considered equally flexible, while the PE captures the idea of an optimal decision given a reference point when the reference point is less flexible. In particular, the PE avoids that a choice is called optimal just because it induces a reference point that is favorable for any decision. However, in a PE that is not an CPE, the decision maker does not achieve his maximum utility. For a further discussion and comparison of the PE and the CPE, see Belica and Ehrhart (2013). Given the continuing binary comparisons that a bidder's decision has to survive during the course of an auction, we consider the PE, which has a fixed reference point, and its refinements more suitable for our analysis.

We need to define PE for the EA and the DA separately, due to the inverse price dynamics of the auctions.

Definition 1 (Personal Equilibrium (PE)). *Given a value v and the beliefs $H(\cdot)$ or $G(\cdot)$, b^* is a Personal Equilibrium (PE)*

- *in the EA if* $U(A(p,b^*)|A(p,b^*)) \geq U(A(p,b)|A(p,b^*))$ *for all* b *and* $p \leq \min\{b^*, b\}$,
- *in the DA if* $U(W(p,b^*)|W(p,b^*)) \geq U(W(p,b)|W(p,b^*))$ *for all* b *and* $p \geq \max\{b^*, b\}$.

That is, in the EA, taking the plan to approve all prices up to b^* as given, at each price $p \leq b^*$ no other bidding limit b appears more attractive than b^* to the bidder. In the DA, taking the plan to wait until b^* as given, there is at each price $p > b^*$ no other plan b that appears more attractive than b^* to the bidder.

What bid does a bidder choose if he compares PE across reference points? That is, what is the PPE? A PPE should be consistent in the sense that a player follows one bid through (i.e. the PPE is a PE). Given his beliefs about the others, a PPE is a bidder's best PE throughout the auction (i.e. for all p). Again, we need to define PPE for the EA and the DA separately, due to the inverse price dynamics of the auctions.

Definition 2 (Preferred Personal Equilibrium (PPE)). *Given a value v and the beliefs $H(\cdot)$ or $G(\cdot)$, b^{**} is a Preferred Personal Equilibrium (PPE)*

- *in the EA, if b^{**} is a PE and if*
  - (1) *for all PE* $b \neq b^{**}$: $U(A(p,b^{**})|A(p,b^{**})) \geq U(A(p,b)|A(p,b))$ *for all* $p \leq \min\{b, b^{**}\}$
  - (2a) *for all PE* $b < b^{**}$: $U(A(p,b^{**})|A(p,b^{**})) \geq U(A(b,b)|A(b,b))$ *for all* $b < p \leq b^{**}$
  - (2b) *for all PE* $b > b^{**}$: $U(A(b^{**},b^{**})|A(b^{**},b^{**})) \geq U(A(p,b)|A(p,b))$ *for all* $b^{**} < p \leq b$
- *in the DA, if b^{**} is a PE and if*
  - (1) *for all PE* $b \neq b^{**}$: $U(W(p,b^{**})|W(p,b^{**})) \geq U(W(p,b)|W(p,b))$ *for all* $p \geq \max\{b, b^{**}\}$
  - (2a) *for all PE* $b < b^{**}$: $U(W(b^{**},b^{**})|W(b^{**},b^{**})) \geq U(W(p,b)|W(p,b))$ *for all* $b \leq p < b^{**}$
  - (2b) *for all PE* $b > b^{**}$: $U(W(p,b^{**})|W(p,b^{**})) \geq U(W(b,b)|W(b,b))$ *for all* $b^{**} \leq p < b$

The difficulty in defining the PPE stems from the dynamics of the auction. At different p, different PE may seem best. A player who anticipates the future knows that the best PE at p might not be his best choice later on, that is, he knows that he will not follow this PE through but he will prefer to switch to another PE. Thus, a PE that maximizes utility over all PE at every p, i.e. a PPE, might not exist.

2.7. Mutual Best Responses: Personal Equilibrium Profiles

We now switch from the point of view of an individual bidder, who holds arbitrary beliefs about the others, to strategic interaction, where beliefs are required to be derived from strategies and value distributions F, and where we need to specify complete bidding functions $\beta_i(v_i)$ for all $i \in N$. In what follows, whenever we consider one representative bidder we skip the index i.

Assumption 1. *For any v and $H(\cdot)$ there exists a $p > \max\left\{v\frac{1+\gamma_g}{1+\lambda_m}, v\frac{1+\lambda_g+\gamma_g}{1+\lambda_m+\gamma_m}\right\}$ such that $H(\cdot)$ has full support on $[0,p]$.*

The assumption on values, $v \in [0,\bar{v})$, implies Assumption 1, and Assumption 1 simplifies our analysis and presentation. With these assumptions, a bidder assigns positive probability to any bid and, in particular, he is never sure about having the highest value and he never beliefs to submit the highest bid with certainty.

In the analysis of the DA, more details on beliefs will be relevant.

Assumption 2. *For any v and $G(\cdot)$ there exists a $\bar{p}$ such that $G(\bar{p}) = 1$.*

Throughout the analysis of PE profiles in the DA we assume a symmetric model and solve for symmetric bidding equilibria. Let $\bar{F}(\cdot)$ and $\bar{f}(\cdot)$ denote the distribution and density of the other $n-1$ bidders' highest value, i.e. $\bar{F}(x) = F(x)^{n-1}$ and $\bar{f}(\cdot) = (n-1)f(x)F(x)^{n-2}$. We assume that $\bar{F} : [0,\bar{v}) \to [0,1]$ is continuous and strictly increasing on $[0,\bar{v})$. This is for example the case if the cdf of the values, F, is continuous and strictly increasing on $[0,\bar{v})$.

Throughout the analysis of the DA, we will use the following monotonicity assumption to characterize PE of a single bidder and to derive symmetric PE profiles.

Assumption 3. *The strategy profile is such that, for any bidder, the function $k : [0,\bar{p}] \to \mathbb{R}_+$ with $k(p) := p + \frac{G(p)}{g(p)}$ is weakly increasing and continuous.*

Lemma 4 in Appendix D justifies Assumption 3 for symmetric PE profiles by showing that $k(p) = p + \frac{G(p)}{g(p)}$ is increasing and continuous for a bidder, if the $n-1$ other bidders bid according to $\beta(v) = a\hat{\beta}(v)$ with $a > 0$ and

$$\hat{\beta}(v) := \frac{\int_0^v s\bar{f}(s)\mathrm{d}s}{\bar{F}(v)} = v - \frac{\int_0^v \bar{F}(s)\mathrm{d}s}{\bar{F}(v)}.$$

Remember that $\hat{\beta}(v)$ is the symmetric equilibrium bidding function of the DA with loss-neutral and risk-neutral bidders, where a bidder bids the expected highest valuation of the other bidders conditional on this value being below his own valuation v.

Let us consider solution concepts for the auction game. What if we require that all bidders choose PE and that beliefs are consistent with these PE? The answer are the PE profiles. Let $\beta_{-i}(v_{-i}) := (\beta_1(v_1), \ldots, \beta_{i-1}(v_{i-1}), \beta_{i+1}(v_{i+1}), \ldots, \beta_n(v_n))$ and remember Definition 1 of PE.

Definition 3 (Personal Equilibrium Profile (PE Profile)). *A profile $(\beta_1(v_1), \ldots, \beta_n(v_n))$ is a personal equilibrium profile (PE profile) in the EA or in the DA, if, for each $i \in N$ and each v_i, and for beliefs $H(\cdot)$ or $G(\cdot)$ that are derived from $\beta_{-i}(v_{-i})$, $\beta_i(v_i)$ is a PE.*

We call a profile $(\beta_1(v_1), \ldots, \beta_n(v_n))$ *symmetric*, if all bidders use the same bidding strategy $\beta(v) = \beta_1(v_1) = \ldots = \beta_n(v_n))$.

Note that for gain-loss neutral bidders the PE profile is equivalent to the Bayes-Nash equilibrium.

Definition 4 (Preferred Personal Equilibrium Profile (PPE Profile)). *A profile $(\beta_1(v_1), \ldots, \beta_n(v_n))$ of bidding functions is a preferred personal equilibrium profile (PPE profile) if $(\beta_1(v_1), \ldots, \beta_n(v_n))$ is a PE profile and if $\beta_i(v_i)$ is a PPE for any i and v_i.*

That is, a PPE profile is a PE profile in which every bidder chooses a PPE.

3. Equilibrium Analysis

The proofs of all propositions in this section are given in Appendix C for the EA and in Appendix D for the DA.

3.1. Example

Figure 1 illustrates how utilities develop during the auction in a symmetric setting for a bidder with valuation $v = 0.7$ (and $n = 2$, $v \sim U[0, 1)$, $\lambda_g = \lambda_m = 0.2$, $\gamma_g = \gamma_m = 0.1$). We plot the utility from the bid $\beta(v)$ and that from the alternative (to quit if $p < \beta(v)$ or quitting only at p if $p > \beta(v)$ in the EA, and to bid if $p > \beta(v)$ or bidding only at p if $p < \beta(v)$ in the DA) under the reference bid $\beta(v)$ for different price levels (starting from zero in the EA and starting from $\bar{p} = 1$ with $G(\bar{p}) = 1$ in the DA). The dashed line with the short dashes is the utility from $\beta(v)$. The dashed line with the long dashes gives the utility from the alternative. We consider a bidder in the earliest reached PE profile, that is, in the lowest PE profile in the English auction and in the highest PE profile in the Dutch auction. The solid line shows that the difference between the two utilities is always positive, that is, $\beta(v)$ provides higher utility than the alternative, which of course has to hold in any PE. Also, it needs to hold that the difference is zero at $p = \beta(v)$, when the bidder quits in the EA or bids in the DA.

In the EA, the utility from $\beta(v)$ is positive and monotonically decreases to zero, which is reached at the PE profile bid 0.64.[12] The utility from the alternative is always negative, with a maximum of zero at 0.64. His utility from winning with $\beta(v)$ is absolutely and relatively to the alternative decreasing the closer the time to quit.

In the DA, the utility from $\beta(v)$ is positive and starts to increase at 0.55, the bid of a bidder whose value is almost 1 in the symmetric PE profile. The utility from the alternative starts out negative, becomes positive at 0.64 and increases until the PE profile bid 0.38 is reached.

Thus, in the DA in contrast to the EA the bidder's expected utility from the bid $\beta(v)$ increases during the course of the auction. Note that the same would hold true for a loss-neutral bidder. In the EA, the conditional probability of winning decreases in p and prices also increase, and, thus, his expected utility decreases and vanishes when he quits. In the in the DA, the probability of winning with $\beta(v)$ increases when p decreases and the payment in case of winning is fixed. With our parameters, in the EA, the utility of a loss-averse bidder is for all $p \leq 0.7$ below that of a loss neutral bidder (who would bid 0.7), due to the loss-utility from losing the money, which is not overcompensated by the gain-utility from gaining the good. In the DA, a loss-averse bidder would bid higher than a loss-neutral bidder, who would bid 0.35 both in the symmetric equilibrium with loss-neutral bidders and as a best response to the equilibrium bids of the loss-averse bidders. The

[12] For PE bids above the minimum PE, the utility from $\beta(v)$ would become negative for p close to $\beta(v)$, but the bidder would approve because the alternative – quitting – would provide more negative utility.

loss-averse bidder's utility at any $p \geq \beta(v)$ is lower than that of a loss-neutral bidder with the same bid, because of his loss-utility from losing the good, which is not overcompensated by his gain-utility from winning money.[13] He compensates this 'disadvantage' by a more aggressive, higher bid, which increases the probability of winning, and, thus, decreases the probability of losing the good.

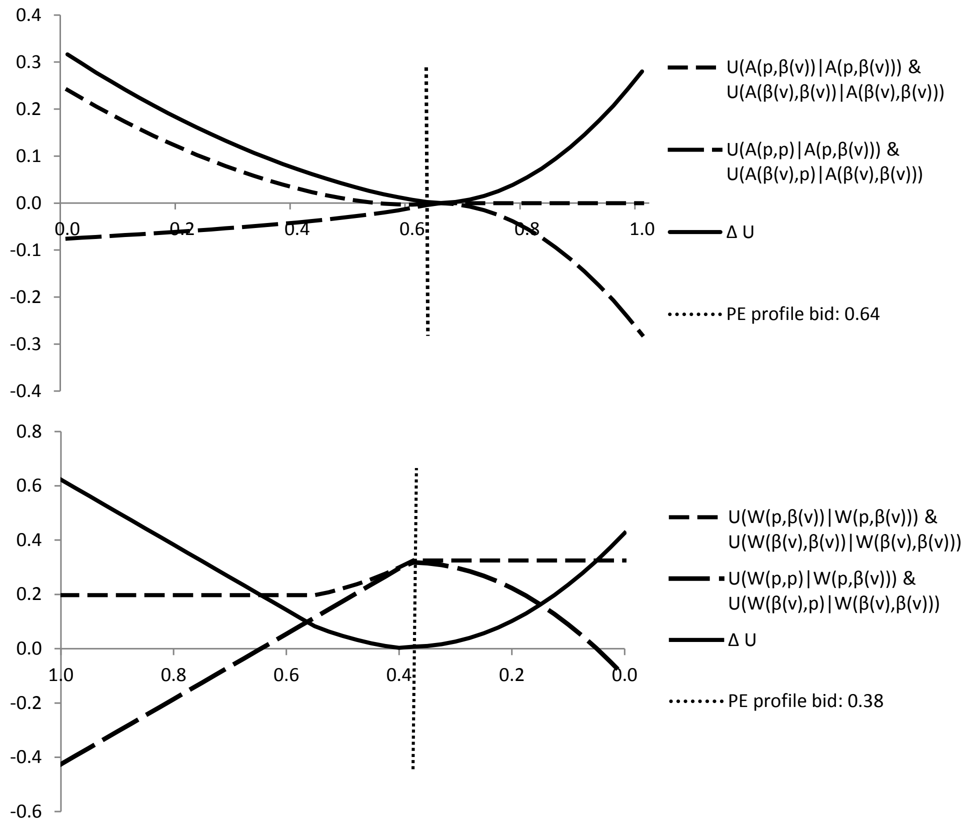

Figure 1: A bidder's utilities from his two options and the difference between the two in a symmetric PE profile during the course of the auction when the price increases from zero to one in the English Auction (upper plot) or decreases from one to zero in the Dutch Auction (lower plot) for $n = 2$, $v \sim U[0,1)$, $v = 0.7$, $\lambda_g = \lambda_m = 0.2$, $\gamma_g = \gamma_m = 0.1$.

[13]For a loss-neutral bidder, the equivalent to the dashed lines with the long dashes would be a horizontal line at zero in the EA (zero utility in case of quitting) and the function $0.7 - p$ for $p \geq 0.35$ (the utility $0.7 - p$ in case of bidding at p) and a horizontal line at 0.35 for $p \leq 0.35$ (his utility from having bid at 0.35) in the DA.

3.2. English Auction

3.2.1. Personal Equilibria

A bid b^* is a PE, if, taking the plan to bid up to b^* as given, at each price $p \leq b^*$ no other bidding limit b appears more attractive to the bidder. This condition is equivalent to two intuitive conditions that emphasize the binary bidding character.

Proposition 1. *Given a value v and the beliefs $H(\cdot)$, b^* is a PE in the EA if and only if*

(EA1) $U(A(p, b^*)|A(p, b^*)) \geq U(A(p, p)|A(p, b^*))$ *for all* $p \leq b^*$ *and*

(EA2) $U(A(b^*, b^*)|A(b^*, b^*)) \geq U(A(b^*, b)|A(b^*, b^*))$ *for all* $b > b^*$.

Proposition 1 is rooted in the construction of our utility function. The bidders anticipate the course of the auction and, thus, revisions of choices are not necessary. A bid that is less attractive when it is reached than the chosen bid seemed less attractive from the beginning, and the same holds the other way round.

The next proposition considers a range of bids that are a PE independent of the beliefs about the others. Thus, bids in this range may be considered dominant strategies given the respective reference point.

Proposition 2. *For a bidder with the value v in the EA,*

(a) *a PE exists if and only if* $\gamma_m(1 + \gamma_g) \leq \lambda_g(1 + \lambda_m)$,

(b) *for every PE b^* it holds that*

$$b^* \in \left[v\frac{1 + \gamma_g}{1 + \lambda_m}, v\frac{1 + \lambda_g + \gamma_g}{1 + \lambda_m + \gamma_m}\right],$$

(c) *b^* is a PE for any beliefs $H(\cdot)$ if and only if* $\gamma_m(1 + \gamma_g) \leq \lambda_g(1 + \lambda_m)$ *and*

$$b^* \in \left[v\frac{1 + \gamma_g}{1 + \lambda_m}, v \min\left\{\frac{1 + \lambda_g + \gamma_g}{1 + \lambda_m + \gamma_m}, \frac{1 + \lambda_g}{1 + \gamma_m}\right\}\right],$$

(d) *for any $H(\cdot)$, if b^* is a PE, every $b \in \left[v\frac{1+\gamma_g}{1+\lambda_m}, b^*\right)$ is also a PE.*

Part (a) characterizes existence, part (b) defines the range of possible PE, part (c) characterizes the interval of belief-independent PE, which always exist if any PE exists, and part (d) describes a continuity property of the range of PE. Note that the existence condition in (a) is equivalent to an existence condition for the interval in (b) (see (C.3)).

Conditions (EA1) and (EA2) differ with respect to the requirements on bidder's beliefs $H(\cdot)$. According to (C.2), Condition (EA2) requires b^* to be at least $v\frac{1+\gamma_g}{1+\lambda_m} = \underline{b}$ whenever $H(\cdot)$ has positive probability mass on $[b, \underline{b}]$ for some $b < \underline{b}$. Thus, there is no PE with bids below $\underline{b}$. In contrast, according to (C.1), Condition (EA1) allows, for specific $H(\cdot)$, for b^* above the upper bound $\bar{b}$, if $\bar{b} = \frac{1+\lambda_g+\gamma_g}{1+\lambda_m+\gamma_m}v < \frac{1+\lambda_g}{1+\gamma_m}v$ and even if $H(\cdot)$ has positive probability mass above $\bar{b}$. Accordingly, there may exist further PE with higher bids than $\bar{b}$ that depend on $H(\cdot)$.

Depending on his the gain-loss parameters, a bidder has only belief-free PE (if $\frac{1+\gamma_g}{1+\lambda_m} \leq \frac{1+\lambda_g+\gamma_g}{1+\lambda_m+\gamma_m} \leq \frac{1+\lambda_g}{1+\gamma_m}$), belief-free and potentially also belief-dependent PE (if $\frac{1+\gamma_g}{1+\lambda_m} \leq \frac{1+\lambda_g}{1+\gamma_m} \leq \frac{1+\lambda_g+\gamma_g}{1+\lambda_m+\gamma_m}$), or he has no PE (if $\frac{1+\gamma_g}{1+\lambda_m} > \frac{1+\lambda_g+\gamma_g}{1+\lambda_m+\gamma_m} \iff \gamma_m(1+\gamma_g) > \lambda_g(1+\lambda_m)$). Only belief-free PE exist for example for a bidder that has the same gain-loss sensations for the good as for money ($\lambda_m = \lambda_g$ and $\gamma_m = \gamma_g$). Both belief-free and belief-dependent PE may for example occur if the bidder is gain-loss neutral with respect to money but not with respect to the good ($\lambda_m = \gamma_m = 0$ and $\lambda_g > \gamma_g \geq 0$). A bidder that is gain-loss neutral with respect to the good but not with respect to money does never have a PE ($\lambda_g = \gamma_g = 0$ and $\lambda_m \geq \gamma_m > 0$).

What bidding strategy does a bidder choose if he compares PE across reference points? That is, what are the PPE? In general PPE might not exist, but a PPE exists if $\lambda_m = 0$, which implies $\gamma_m = 0$, that is, the bidder is gain-loss neutral with respect to money, and which implies existence of PE.

Proposition 3. *Given a value v and the beliefs $H(\cdot)$, if $\lambda_m = 0$, a PPE exists and $b^{**} = v(1+\gamma_g)$ is his unique PPE.*

3.2.2. Personal Equilibrium Profiles

Now let us consider solutions to the auction game. What if we require that all bidders choose PE and that beliefs are consistent with these PE; that is, what are the PE profiles?

Corollary 1. *In the EA, if $\gamma_m(1+\gamma_g) \leq \lambda_g(1+\lambda_m)$, any $(\beta_1(v_1), \beta_2(v_2), \ldots, \beta_n(v_n))$ with $\beta_i(v_i) \in \left[v_i\frac{1+\gamma_g}{1+\lambda_m}, v_i \min\left\{\frac{1+\lambda_g+\gamma_g}{1+\lambda_m+\gamma_m}, \frac{1+\lambda_g}{1+\gamma_m}\right\}\right]$ for every $i \in N$ is a PE profile.*

This follows directly from Proposition 2(c). Note that these PE profiles need not be symmetric. Also, a bidder's bid depends only on his own parameter values and on his own value for the good. In that sense, the PE profiles have similar properties as the Nash equilibrium in undominated strategies with gain-loss-neutral bidders (to which the set of PE profiles reduces if all parameters are zero).

The PPE profile follows directly from Proposition 3 if $\lambda_m = 0$ because the PPE is the same for any beliefs $H(\cdot)$.

Corollary 2. *In the EA, $(\beta_1(v_1), \beta_2(v_2), \ldots, \beta_n(v_n))$ with $\beta_i(v_i) = v_i \frac{1+\gamma_g}{1+\lambda_m}$ for every $i \in N$ is the unique PPE profile if $\lambda_m = 0$.*

3.3. Dutch Auction

3.3.1. Personal Equilibria

A bid b^* is a PE if, taking the plan to wait until b^* as given, there is at each price $p > b^*$ no other plan b that appears more attractive to the bidder. This condition is equivalent to two intuitive conditions, that emphasize the binary bidding character.

Proposition 4. *Given a value v and the beliefs $G(\cdot)$, b^* is a PE in the DA if and only if*

(DA1) $U(W(p, b^*)|W(p, b^*)) \geq U(W(p, p)|W(p, b^*))$ *for all $p \geq b^*$ and*

(DA2) $U(W(b^*, b^*)|W(b^*, b^*)) \geq U(W(b^*, b)|W(b^*, b^*))$ *for all $b \leq b^*$.*

For a bidder with value v the question behind (DA1) is ‚Should I bid now or wait until b^* (given my reference bid b^*)?,' and the question behind (DA2) is ‚Is bidding now at b^* a good plan or is it better to wait (given my reference bid b^*)?' If we could find a range of b^* for that the answer to the first question is 'yes,' and another range of b^* for that the answer to the second question is 'yes,' then all b^* in the intersection of the two ranges would all be PE.

In our analysis we will use the following Lemma, which uses the simplified notation

$$D_1(p, b, b^*) := U(W(p, b^*)|W(p, b^*)) - U(W(p, b)|W(p, b^*)) \text{ with } p \geq b \geq b^* \tag{14}$$

$$D_2(p, b, b^*) := U(W(p, b^*)|W(p, b^*)) - U(W(p, b)|W(p, b^*)) \text{ with } b \leq b^* \leq p. \tag{15}$$

Note that (DA1) equals $D_1(p, p, b^*) \geq 0$ for all $p \geq b^*$ and (DA2) is $D_2(b^*, b, b^*) \geq 0$ for all $b \leq b^*$. Thus, any condition on $D_1(\cdot)$ and $D_2(\cdot)$ will relate to (DA1) and (DA2), respectively. The following Lemma gives necessary and sufficient conditions for (DA1) and (DA2) to hold.

Lemma 1. *Given a value v and the beliefs $G(\cdot)$, b^* is a PE*

- *only if* $\frac{\partial D_1(p,b,b^*)}{\partial b} \geq 0$ *for* $b = p = b^*$ *and* $\frac{\partial D_2(b^*,b,b^*)}{\partial b} \leq 0$ *for* $b = b^*$
- *if* $\frac{\partial D_1(p,b,b^*)}{\partial b} \geq 0$ *for all* p *and* b *with* $p \geq b \geq b^*$ *and* $\frac{\partial D_2(b^*,b,b^*)}{\partial b} \leq 0$ *for all* $b \leq b^*$.

Lemma 1 gives the necessary conditions for b^* being a PE

$$\left((1+\gamma_g+\lambda_g)v - (1+\gamma_m+\lambda_m)b^*\right)g(b^*) - (1+\lambda_m)G(b^*) \leq 0 \tag{16}$$

$$((1+\lambda_g)v - (1+\gamma_m)b^*)g(b^*) - (1+\gamma_m)G(b^*) \geq 0, \tag{17}$$

and the sufficient conditions

$$\left((1+\gamma_g+\lambda_g G(b^*|b))\, v - (1+\lambda_m)\, b - \gamma_m b^* G(b^*|b)\right) g(b) - (1+\lambda_m)G(b) \leq 0 \text{ for all } p \geq b \geq b^*$$
$$((1+\lambda_g)v - (1+\gamma_m)b)g(b) - (1+\gamma_m)G(b) \geq 0 \text{ for all } b \leq b^*,$$

because

$$\begin{aligned}
\frac{\partial D_1(p,b,b^*)}{\partial b} &= -\frac{\partial U(W(p,b)|W(p,b^*))}{\partial b} \\
&= -\frac{1}{G(p)}\left(\left((1+\gamma_g+\lambda_g G(b^*|b))\, v - (1+\lambda_m)\, b - \gamma_m b^* G(b^*|b)\right) g(b) - (1+\lambda_m)G(b)\right), \\
\frac{\partial D_2(b^*,b,b^*)}{\partial b} &= -\frac{\partial U(W(b^*,b)|W(b^*,b^*))}{\partial b} = -\frac{1}{G(b^*)}\left(((1+\lambda_g)v - (1+\gamma_m)b)g(b) - (1+\gamma_m)G(b)\right).
\end{aligned}$$

Proposition 5. *Given a value v and the beliefs $G(\cdot)$, it holds that*

(a) b^* *is a PE* $\implies \frac{(1+\lambda_g)v-(1+\gamma_m)\frac{G(b^*)}{g(b^*)}}{1+\gamma_m} \geq b^* \geq \frac{(1+\gamma_g+\lambda_g)v-(1+\lambda_m)\frac{G(b^*)}{g(b^*)}}{1+\gamma_m+\lambda_m}$

(b) $\frac{(1+\lambda_g)v-(1+\gamma_m)\frac{G(b^*)}{g(b^*)}}{1+\gamma_m} \geq b^* \geq \frac{(1+\gamma_g+\lambda_g)v-(1+\lambda_m)\frac{G(b^*)}{g(b^*)}}{1+\lambda_m} \implies b^*$ *is a PE*

(c) *if* $\lambda_g \geq \gamma_m$, b^* *is a PE* $\iff \frac{(1+\lambda_g)v-(1+\gamma_m)\frac{G(b^*)}{g(b^*)}}{1+\gamma_m} \geq b^* \geq \frac{(1+\gamma_g+\lambda_g)v-(1+\lambda_m)\frac{G(b^*)}{g(b^*)}}{1+\gamma_m+\lambda_m}$

Condition (16), which is derived from (DA1), determines the lower bound on PE while condition (17), which is derived from (DA2), determines the upper bound. This is in line with the intuition on (DA1) and (DA2) given above. However, not all b^* in this intersection of the ranges for that (16) and (17) hold, respectively, need to be PE. While (17) is necessary and sufficient for the upper bound, this does not hold for (16) and the lower bound. Our sufficient lower bound might not be tight and PE b^* with $\frac{(1+\gamma_g+\lambda_g)v-(1+\lambda_m)\frac{G(b^*)}{g(b^*)}}{1+\lambda_m} \geq b^* \geq \frac{(1+\gamma_g+\lambda_g)v-(1+\lambda_m)\frac{G(b^*)}{g(b^*)}}{1+\gamma_m+\lambda_m}$ may exist. However, according to (c), if $\lambda_g \geq \gamma_m$, this sufficient condition is also necessary.

The existence of PE depends on the gain-loss parameters.

Proposition 6. *Given beliefs $G(\cdot)$ it holds that*

(a) *if $(\lambda_m - \gamma_m)(1+\lambda_g) \geq \gamma_g(1+\gamma_m)$, then a PE exists,*

(b) *if $\lambda_m(1+\lambda_g) < \gamma_g(1+\gamma_m)$, then a PE does not exist.*

For bidders that are gain-neutral in the good ($\gamma_g = 0$), a PE exists. For $\lambda := \lambda_g = \lambda_m$ and $\gamma := \gamma_g = \gamma_m$ the sufficient condition holds at least for all $\lambda \geq 2\gamma$. A PE does not exist for example for a bidder that is gain-loss-neutral with respect to money but not gain-neutral with respect to the good ($\lambda_m = \gamma_m = 0$, $\lambda_g \geq \gamma_g > 0$).

Note that the conditions in Proposition 5 have to be checked for every b^* independently, because the boundaries depend on the bid b^* itself. However, if a PE b^* exists, we can immediately conclude on other PE, and in particular on the maximum PE given the beliefs $G(\cdot)$.

Lemma 2. *Given a value v and the beliefs $G(\cdot)$, if b^* is a PE, then a unique maximum PE $\bar{b} = \frac{(1+\lambda_g)v-(1+\gamma_m)\frac{G(\bar{b})}{g(\bar{b})}}{1+\gamma_m} \geq b^*$ exists and every $b \in [b^*, \bar{b}]$ is a PE.*

Corollary 3. *If b^* is a PE, then there exists exactly one interval I of PE with $I = [\underline{b}, \bar{b}]$ and $\underline{b} \leq b^*$.*

Now that we solved for PE, we ask what bid a bidder chooses if he compares PE across reference points, that is, what is the PPE of a DA? In the DA, PE that appear best throughout the auction usually do not exist. Thus, we cannot characterize PPE. It holds that at any PE, if it is reached by the auction price, lower PE appear less attractive.

3.3.2. Personal Equilibrium Profiles

Next we will address PE profiles $(\beta_1(v), \ldots, \beta_n(v))$ with a focus on *symmetric PE profiles*, $\beta(v) := \beta_1(v) = \cdots = \beta_n(v)$. A profile $(\beta_1(v), \ldots, \beta_n(v))$ is a PE profile if and only if first, beliefs

$G(\cdot)$ are derived from the profile and the distributions of values, and second, $\beta_i(v)$ is a PE for every v and every i. Throughout this section we assume symmetric bidders.

Proposition 7. *If and only if $(\lambda_m - \gamma_m)(1+\lambda_g) \geq \gamma_g(1+\gamma_m)$ there exist symmetric PE profiles with $\beta(v) \in [\underline{\beta}(v), \bar{\beta}(v)]$ and with the monotone interval boundaries*

$$\underline{\beta}(v) = \frac{1+\gamma_g+\lambda_g}{1+\lambda_m}\,\hat{\beta}(v) \quad \textit{and} \quad \bar{\beta}(v) = \frac{1+\lambda_g}{1+\gamma_m}\,\hat{\beta}(v).$$

Moreover, given v, for any $\beta(v) \in [\underline{\beta}(v), \bar{\beta}(v)]$ there exists a symmetric PE profile in which $\beta(v)$ is chosen.

Proposition 7 mainly shows existence of a continuum of symmetric PE profiles for a range of parameter values. Can there be other symmetric PE profiles? With respect to the upper bound the answer is unambiguous.

Proposition 8. *In the DA, if a symmetric PE profile $(\beta(v), \ldots, \beta(v))$ exists, then $\beta(v) \leq \bar{\beta}(v)$.*

Depending on the value of the parameter γ_m, that is, in how far a bidder considers not winning the auction as gaining money, we can be more specific about low-bid equilibria. For $\gamma_m = 0$, if a symmetric PE profile exists, $\underline{\beta}(v)$ constitutes the smallest symmetric PE profile. For this parameter value, the necessary and sufficient conditions for a PE in Proposition 5 are identical and the claim follows from the proof of Proposition 7.

For $\gamma_m > 0$, symmetric PE profiles smaller than $\underline{\beta}(v)$ may exist.

Lemma 3. *In the DA, the smallest symmetric PE profile that can exist is constituted by the monotonic bidding function*

$$\beta_{min}(v) = a\frac{\int_0^v s\bar{f}(s)\bar{F}(s)^{c-1}\,ds}{\bar{F}(v)^c} = \frac{a}{c}\left(v - \frac{\int_0^v \bar{F}(s)^c ds}{\bar{F}(v)^c}\right)$$
$$\textit{with } a = \frac{1+\lambda_g+\gamma_g}{1+\lambda_m} \textit{ and } c = \frac{1+\lambda_m+\gamma_m}{1+\lambda_m}$$

One can show that $\beta_{min}(v)$ constitutes a symmetric PE profile, if $\lambda_g \geq \gamma_m > 0$ and $v\bar{f}(v)/\bar{F}(v) \leq (1+\lambda_m)/\gamma_m$ for all $v > 0$. Then an interval of symmetric PE profiles exists, where $\beta_{min}(v)$ determines the lower bound and $\bar{\beta}(v)$ determines the upper bound. This constitutes the largest possible interval of symmetric PE profiles.[14] The symmetric PE profiles that cover the interval can be generated by

$$
\begin{aligned}
&\beta_{min}(v) = a\,\frac{\int_0^v s\bar{f}(s)\bar{F}(s)^{c-1}\,\mathrm{d}s}{\bar{F}(v)^c} \text{ with } a = \frac{1+\lambda_g+\gamma_g}{1+\lambda_m} \text{ and } c = \frac{1+\lambda_m+\gamma_m}{1+\lambda_m} \\
&< \beta(v) = a\,\frac{\int_0^v s\bar{f}(s)\bar{F}(s)^{c-1}\,\mathrm{d}s}{\bar{F}(v)^c} \text{ with } a = \frac{1+\lambda_g+\gamma_g}{1+\lambda_m} \text{ and } c \in \left(1, \frac{1+\lambda_m+\gamma_m}{1+\lambda_m}\right) \\
&< \beta(v) = a\,\frac{\int_0^v s\bar{f}(s)\,\mathrm{d}s}{\bar{F}(v)} \text{ with } a \in \left[\frac{1+\lambda_g+\gamma_g}{1+\lambda_m}, \frac{1+\lambda_g}{1+\gamma_m}\right].
\end{aligned}
$$

Proposition 9. *In the DA with $\lambda_m = \gamma_m = 0$, a PPE profile exists if $\gamma_g = 0$ and consists of $\beta^*(v) = (1+\lambda_g)\hat{\beta}(v)$.*

The result is obvious, because our necessary and sufficient conditions for existence of PE in Proposition 6 coincide and equal $0 \geq \gamma_g$, $\beta^*(v)$ constitutes a symmetric PE by Proposition 7, and the corresponding bids are the only PE of each bidder by Proposition 5.

4. Discussion

We discuss differences in the influence of the parameters on bidding in the EA and the DA and provide intuition on the sources of these differences. For specific popular parameter regions we specify our results. We argue that in laboratory experiments with the EA and induced values, loss averse bidders bid v in any PE. Then, we compare the auctions' expected revenues.

4.1. The Influence of the Gain-Loss Parameters on Bidding in the EA and in the DA

For discussing differences between the auctions we focus on the symmetric PE profiles that we know to exist if any PE profile exists. By Proposition 2 these are for the EA the PE profiles with

$$\beta^*(v) \in \left[v\frac{1+\gamma_g}{1+\lambda_m}, v\,\min\left\{\frac{1+\lambda_g+\gamma_g}{1+\lambda_m+\gamma_m}, \frac{1+\lambda_g}{1+\gamma_m}\right\}\right],$$

and by Proposition 7 these are for the DA the PE profiles with

$$\beta^*(v) \in \left[\frac{1+\gamma_g+\lambda_g}{1+\lambda_m}\,\hat{\beta}(v), \frac{1+\lambda_g}{1+\gamma_m}\,\hat{\beta}(v)\right].$$

[14] The proofs of the claims in this paragraph are available from the authors upon request.

We concentrate on the loss parameters and consider the gain parameters as potentially mitigating the effects of the loss parameters, because a gain parameter never appears in these boundaries without the loss parameter of the other consumption dimension. In general and quite in line with intuition, the good-loss parameter tends to increase bids while the money-loss parameter tends to decrease bids. The good-loss parameter appears in the upper bounds on PE profiles in both auctions, while the money-loss parameter appears in the lower bound in the EA and in the low PE profile bids given above for the DA. Losing the good seems a mayor concern in the DA while losing money is more important in the EA. In the EA, the good-loss parameter appears in at most one bound (the upper bound), while the money-loss parameter appears in the lower bound and may appear in the upper bound. In the DA, the money-loss parameter appears only in the the lower bound while the good-loss parameter appears in both bounds. Intuitively, in the EA the bid determines when to give up the good, so losing the good is no major concern in this auction, while in the DA the bid determines what to pay for sure, so losing money is less important than someone else taking the good by bidding earlier.

In the EA, for answering the question behind (EA1), 'Is approving until b better than quitting now?', both λ_m and λ_g may play a role. For a PE, the answer must be 'yes' while $b^* = b$ is in the future and, thus, sensations of losing the good are involved with the alternative decision, whereas sensations of losing money are evoked by the decision to approve (cp. equations (5) and (6)). (EA1) determines the highest PE because a b exists that is high enough such that the answer is 'no.' For answering the question behind (EA2), 'Is quitting now at b better than increasing the bid?' only λ_m plays a role. When b is reached, the PE answer is 'yes,' the bidder quits, and his utility is zero, whereas further approving would involve sensations of losing the good (cp. equations (7) and (8)). (EA2) determines the lowest PE, because a sufficiently low b such that the answer is 'no' exists. Thus, (EA1) limits the effect of λ_g while (EA2) limits the effect of λ_m.

In the DA, for answering the question behind (DA1), 'Is waiting until b better than bidding now?', both λ_g and λ_m play a role. For a PE the answer must be 'yes' while $b^* = b$ is in the future and, thus, sensations of losing money are involved with instead bidding now, while sensations of losing the good come with waiting (cp. (10) and (11)). (DA1) determines the lowest PE because a p exists that is low enough such that the answer is 'no' for all b. For answering the question behind (DA2), 'Is bidding now at b better than waiting longer?', only λ_g plays a role. When b

is reached, the PE answer is 'yes,' the bidder bids, and his utility is $v - b$, while the alternative would involve sensations of losing the good (cp. (12) and (13)). (DA2) determines the highest PE, because at a sufficiently high b the answer must be 'no.' Thus, (DA1) limits the effect of λ_m while (DA2) limits the effect of λ_g.

Overall, the bid decreasing effect of avoiding sensations of losing money results in a range of symmetric PE in the EA that is shifted downwards relative to the range of symmetric PE profiles in the DA (divided by v and $\beta(v)$, respectively), whose tendency towards higher bids is predominantly driven by avoiding sensations of losing the good.

4.2. Specific Parameter Regions

For a loss-averse bidder, a loss looms larger than a gain of the same size but our parameters $\lambda_k \geq \gamma_k \geq 0$, $k \in \{g, m\}$ allow for gain-loss neutrality. We will discuss our results in particular with respect to parameter regions used in the literature. Remember that in the EA, a PE exists if and only if $\gamma_m(1 + \gamma_g) \leq \lambda_g(1 + \lambda_m)$ by Proposition 2, while in the DA, a PE exists if $(\lambda_m - \gamma_m)(1 + \lambda_g) \geq \gamma_g(1 + \gamma_m)$ and a PE does not exist if $\lambda_m(1 + \lambda_g) < \gamma_g(1 + \gamma_m)$ by Proposition 6. Assuming that the loss sensation for the good is not smaller than for money ($\lambda_g \geq \lambda_m$) is sufficient for a PE in the EA to exist. However, this is not sufficient for the DA, where the existence of PE requires a loss sensation for money ($\lambda_m > 0$) if the bidder has a gain sensation for the good ($\gamma_g > 0$).

Symmetric consumption dimensions: $\lambda_g = \lambda_m$ and $\gamma_g = \gamma_m$. Strictly loss-averse bidders that treat all consumption dimensions equally are probably most popular in the literature. The corresponding parameters are $\lambda_g = \lambda_m =: \eta\zeta$ and $\gamma_g = \gamma_m =: \eta$, with a weight $\eta > 0$ for gain-loss utility and a 'coefficient of loss-aversion' $\zeta > 1$ (Kőszegi and Rabin, 2006).[15] For these parameter values, PE in the EA exist and all PE are belief-free. The range of PE profile bids is $\beta^*(v) \in \left[v\frac{1+\eta}{1+\eta\zeta}, v\right]$. That is, a bidder never bids above his true value but he might want to quit below his value, if his reference bid is below his value. In the DA, PE exist for example for all $\zeta \geq 2$.

Loss-averse symmetric bidders in the DA bid in symmetric PE profiles more than loss-neutral bidders. The symmetric PE profile bids in the range $\beta^*(v) \in \left[\hat{\beta}(v)\frac{1+\eta+\eta\zeta}{1+\eta\zeta}, \hat{\beta}(v)\frac{1+\eta\zeta}{1+\eta}\right]$ are higher

[15] In the literature, what we call ζ is denoted by λ (e.g. Kőszegi and Rabin, 2006).

than $\hat{\beta}(v)$, and also the potentially existing low-bid PE profiles are higher than $\hat{\beta}(v)$.[16]

Gain-neutrality: $\gamma_g = \gamma_m = 0$*.* Gain-neutrality is considered by Lange and Ratan (2010). In this case, PE in the EA exist and all PE are belief-free. The range of PE profile bids is $\beta^*(v) \in \left[v\frac{1}{1+\lambda_m}, v\frac{1+\lambda_g}{1+\lambda_m}\right]$. If the bidder is more loss averse with respect to the good than with respect to money, he might bid above his value.

In the DA, PE exist, and any bid in a symmetric PE profile lies within the range $\beta^*(v) \in \left[\hat{\beta}(v)\frac{1+\lambda_g}{1+\lambda_m}, \hat{\beta}(v)(1+\lambda_g)\right]$. The bid might lie below $\hat{\beta}(v)$ if the bidder is more loss averse with respect to money than with respect to the good. In all symmetric PE profiles, the DA bid divided by $\hat{\beta}(v)$ is weakly higher than the EA bid divided by v.

Focus on the goods dimension: $\gamma_g > \lambda_m$*.* It has been argued that loss-aversion might evoke "auction fever" – bidders bidding above their (initial) values v (see Ariely and Simonson, 2003; Heyman et al., 2004; Wolf et al., 2005; Ehrhart et al., 2013, for arguments based on pseudo-endowment effects and for experimental results with real goods or with non-standard induced values). In our analysis, an auction fever effect arises in the EA for bidders whose sensations of gains and losses with respect to the good are more intense than their sensations in the money dimension. Such bidders with $\gamma_g > \lambda_m$ bid above v in any PE of the EA (and PE in the EA exist). In the DA, PE do not exist if $\gamma_g > \lambda_m$

If we further restrict consideration to $\lambda_m = \gamma_m = 0$, we can compare the EA and (with a further restriction) the DA to the second-price and first-price sealed bid auctions (with results for the sealed-bid auctions taken from Belica and Ehrhart, 2013). In the EA, the PE profile bids are within the range $[v(1+\gamma_g), v(1+\lambda_g)]$ and the PPE profile bid is $v(1+\gamma_g)$. The bid $v(1+\lambda_g\bar{F}(v)+\gamma_g(1-\bar{F}(v)))$ in the PE/PPE profile in the second-price auction is within the range of PE of the EA but higher than the PPE.[17] To compare the auctions to the DA, we have to assume that in addition $\gamma_g = 0$ for existence of PE. Then, the unique symmetric PE profile in

[16] Note that in the shopping shoes example by Kőszegi and Rabin (2006), a buyer would always accept any price below $v\frac{1+\eta}{1+\eta\zeta}$ and the buyer would never accept prices above $v\frac{1+\eta\zeta}{1+\eta}$, which correspond to the lowest equilibrium bid $v\frac{1+\eta}{1+\eta\zeta}$ in the EA and to the highest equilibrium bid $\hat{\beta}(v)\frac{1+\eta\zeta}{1+\eta}$ in the DA.

[17] PE/PPE profile bids are $v(1+\lambda_g\bar{F}(v)+\gamma_g(1-\bar{F}(v)))$ in the second-price auction and $\frac{\int_0^v s(1+\lambda_g\bar{F}(v)+\gamma_g(1-\bar{F}(v)))}{\bar{F}(v)}$ in the first-price auction. These bids are higher than the choice-acclimating personal equilibrium (CPE) bids in both sealed-bid auctions (Belica and Ehrhart, 2013).

the DA consists of $\beta(v) = \hat{\beta}(v)(1+\lambda_g)$, which is higher than the PE bids in the first-price auction. The expected revenue from this unique symmetric PE of the DA equals that from the highest PE profile in the EA. It exceeds the revenue from the second-price auction, which is higher than that from the first-price auction, which in turn is higher than the lowest PE (and PPE) profile revenue in the EA. In all auctions, in line with intuition, relative to a loss-neutral bidder, purely good-loss-averse bidders bid higher.

These results are supported by the field experiment by Lucking-Reiley (1999), who finds a tendency for higher prices in the DA than in the first-price auction. Comparing the EA and the second-price auction, he finds no statistical difference, with some tendency of individuals to bid higher on average in the EA, but heterogeneity across bidders. His results were not in line with experimental results with induced values. We will discuss a possible reason in the next subsection.

4.3. Auction Fever Disappears in the EA in the Laboratory

In laboratory experiments with induced values the auction fever effect described in the previous section (for $\gamma_g > \lambda_m$) may not occur. Following Lange and Ratan (2010), in the standard experimental design, in which participants are assigned a monetary value for the abstract object for sale, the goods dimension does not exist and all evaluations refer to the money dimension (where winning the good is considered winning an amount of money equal to v). If we conduct our analysis of the EA under this assumption, auction fever disappears and the unique PE and, thus, the unique PPE in the EA is $b = v$.[18]

Proposition 10. *Consider bidders that assign the good and money the same consumption dimension, the "monetary rent dimension." A bidder's only PE and thus his unique PPE in the EA is $\beta^*(v) = v$. All bidders choosing $\beta^*(v) = v$ constitutes the unique PE profile and the unique PPE profile.*

The proof is given in Appendix C.

[18] Referring to the separation of consumption dimensions, Kőszegi and Rabin (2006, p. 1138) state that "In combination with loss aversion, this separability is at the crux of many implications of reference-dependent utility, including the endowment effect." Similar to our finding, when focusing on a single monetary rent dimension, Lange and Ratan (2010) show that $b(v) = v$ is the CPE in the second-price auction for sufficiently high v and that bidders with low v do not participate.

Thus, for an auction fever effect, a separation of the money and the goods dimension as well as stronger sensations in the goods dimensions are necessary. This is in line with the experimental results by Ehrhart et al. (2013), who find truthful bidding in all their EAs with induced values, and indications for auction fever in treatments designed to induce separated consumption dimensions and to foster pseudo-endowment and auction fever.

4.4. Comparison of the Revenues in the English and in the Dutch Auctions

To compare the revenues in the EA and in the DA we assume symmetric bidders. Denote the revenues in symmetric PE profiles in the EA and in the DA by R^{EA} and R^{DA} and denote the second order statistics of the private values by $V_{(2:n)}$.

Proposition 11. *Let $\gamma_m(1+\gamma_g) \leq \lambda_g(1+\lambda_m)$ and $(\lambda_m-\gamma_m)(1+\lambda_g) \geq \gamma_g(1+\gamma_m)$, i.e., symmetric PE profiles in the EA and DA exist. Compare expected revenues in symmetric PE profiles.*

(a) *The expected revenue in the EA is lower than the expected revenue in the DA:*

$$\max E[R^{EA}] \leq \min E[R^{DA}].$$

(b) *If $\lambda_m > \gamma_m = 0$, then the highest expected revenue in the EA is equal to the lowest expected revenue in the DA:*

$$\max E[R^{EA}] = \frac{1+\lambda_g+\gamma_g}{1+\lambda_m} E[V_{(2:n)}] = \min E[R^{DA}].$$

If $\lambda_m = \gamma_m = \gamma_g = 0 \leq \lambda_g$, then the expected revenue from the EA is at least as high as with gain-loss neutral bidders and at most as high as that from the unique symmetric PE profile in the DA

$$E[V_{(2:n)}] = \min E[R^{EA}] \leq \max E[R^{EA}] = (1+\lambda_g)E[V_{(2:n)}] = E[R^{DA}]$$

and the revenue in the PPE of the EA is lower than that in the PPE of the DA: $E[V_{(2:n)}] \leq (1+\lambda_g)E[V_{(2:n)}]$.

The proof is given in Appendix E. If symmetric PE profiles exist in both auctions, all PE in the EA are belief-free and the revenues in PE in the EA are never higher than those from symmetric

PE in the DA. Note that if $\lambda_m > \gamma_m = 0$ and $\lambda_m \geq \gamma_g$ then PE exist in both auctions, and this is also true if $\lambda_m = \gamma_m = \gamma_g = 0 \leq \lambda_g$.

5. Conclusion

The EA and the DA evoke quite different sensations of gains and losses in loss-averse bidders during the process of the auction. The opportunity to get the good for sure by submitting a bid in the DA seduces the bidder to earlier, higher bids. Also, sensations of winner and losers are different in the auctions. In the EA, the losing bidders have no sensations of gain or loss due their exit decisions, but the winner feels a loss of money and a gain of the good. In a PE, the winner of the DA has no sensation of gain or loss, but the losers feel a loss of the good and a gain of money. Depending on the feelings towards the auction the seller wants to raise he might prefer the DA with a content winner or the EA with content losing bidders. These differences in sensations predicted by the model might be exploited for experimental testing.

Interestingly, we find ranges of PE profiles. For example, a bidder in the EA with $\lambda_g = \lambda_m = 2\gamma_g = 2\gamma_m = 0.2$ who values the good at 50, may bid anything between 46 and 50 in an equilibrium of the auction, depending on what reference bid he has in mind but independent of his beliefs about the opponents. In symmetric PE profiles of a DA with two bidders, uniformly distributed values on [0,100], and the same gain-loss parameter values, the bidder who values the good at 50 would bid between 26 and 27 (while a gain-loss neutral bidder would bid 25).

An auction fever effect – bidding above v – is predicted for the EA for bidders whose sensations are primarily in the goods dimension. If the bidder in the previous example had no gain-loss sensation in the money dimension, he would bid anything between 55 and 65. If the money and the good consumption dimensions merge to a single monetary rent dimension, this effect disappears and he bids 50. Thus, in laboratory experiments with induced values the effect should disappear, which has been observed.

Expected revenues from symmetric PE profiles in the DA are higher than those from PE profiles in the EA, if PE in both auctions exist. Thus, a seller who wants to maximize his revenue and assumes that a bidder is loss-averse might prefer the DA over the EA.

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Appendix A. Utility Functions

Appendix A.1. English Auction

In the EA, a gain-loss averse bidder's utility function $U(A(p,b)|A(p,\hat{b}))$ if $b \leq \hat{b}$ is:

$$\begin{aligned}
U(A(p,b)|A(p,\hat{b})) &= \int_p^\infty \int_q^\infty u(q|s)h(s|q)h(q|p)\mathrm{d}s\mathrm{d}q \\
&= \int_p^b (v-q)h(q|p)\mathrm{d}q + \gamma_g v \int_p^b h(q|p)(1-H(\hat{b}|q))\mathrm{d}q + \lambda_m \int_p^b -qh(q|p)(1-H(\hat{b}|q))\mathrm{d}q \\
&\quad - \lambda_g v(1-H(b|p)) \int_b^{\hat{b}} h(s|b)\mathrm{d}s + \gamma_m(1-H(b|p)) \int_b^{\hat{b}} sh(s|b)\mathrm{d}s \\
&= \int_p^b \Big(v(1+\gamma_g(1-H(\hat{b}|q))) - q(1+\lambda_m(1-H(\hat{b}|q)))\Big) h(q|p)\mathrm{d}q \\
&\quad - \int_b^{\hat{b}} (\lambda_g v - \gamma_m s)h(s|p)\mathrm{d}s
\end{aligned}$$

Remember, $H(b|p) = \frac{H(b)-H(p)}{1-H(p)}$ and $h(b|p) = \frac{h(b)}{1-H(p)} = (1-H(b|p))h(s|b)$. Figure A.2 visualizes the relevant cases for the derivation of the utility function.

The utility function $U(\cdot|\cdot)$ reflects that the decision maker at p anticipates the course of the auction. The consequences of the bid b are that at any particular stage $q \geq p$ below b the auction continues or he wins (with marginal probability $h(q|p)$), while at any $q > b$ he does not win.

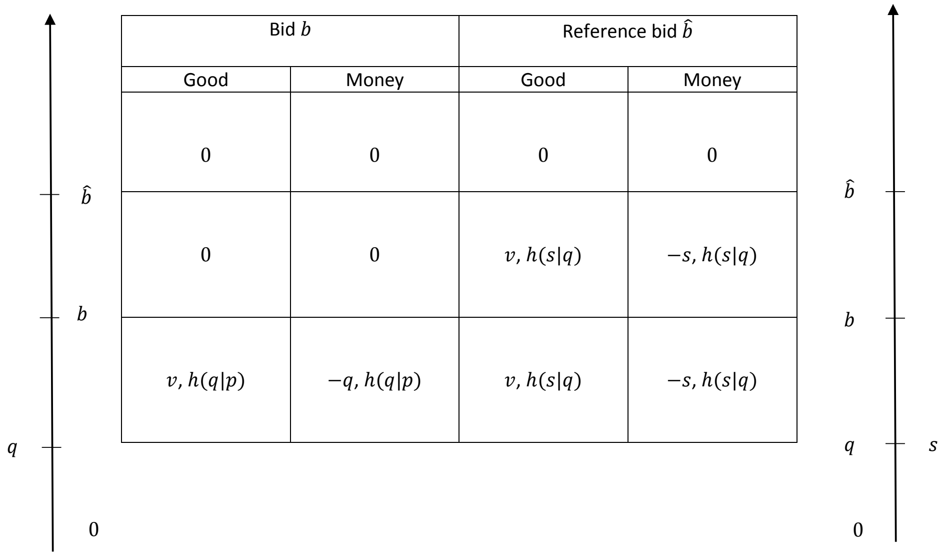

Figure A.2: v or $-q$ $(-s)$ is written in a cell if at stage $q > p$ $(s > q)$ the bid b (the reference bid $\hat{b}$) can be successful such that the auction ends at q (s) with the payment q (s). The other element in the cell is the marginal winning probability $h(q|p)$ $(h(s|q))$ of q (s), given that the auction has reached price level p (q). A zero indicates that the bidder cannot win the auction for the respective values of q or s with b or $\hat{b}$. The variable q takes values larger than p and the variable s takes values larger than q.

His marginal consumption utility at any q, $p \leq q \leq b$, is therefore $h(q|p)(v - q)$. The bidder at p anticipates that given that the auction reaches q, $p \leq q \leq b$, he might have a sensation of gaining the good of value v and losing the amount q of money relative to the reference point, where gains and losses are judged from the point of view q. (That is, the bidder anticipates that he will compare the consequence of b with the reference bid $\hat{b}$, given that q has been reached.) The marginal probabilities of gains v and losses $-q$ are calculated as the product of the marginal probability $h(q|p)$ of winning with b at q seen from p and the marginal probability of not winning with $\hat{b}$, given that q has been reached, which becomes relevant for $s > \hat{b}$ and accumulates to $\int_{\hat{b}}^{\infty} h(s|q)\mathrm{d}s = 1 - H(\hat{b}|q)$. Also, the bidder at p anticipates that if the auction reaches a stage $q > b$, he will have quit without an option to return and win the item. If $q \leq \hat{b}$, the reference bid would then still given him the chance to win the auction, creating sensations of losing the good and winning money, where the marginal probabilities of these gains and losses are calculated as the product of 1 (losing with b) and $h(s|q)$ (the marginal probability of winning the auction with $\hat{b}$) for $s \leq \hat{b}$ and 0 for $s > \hat{b}$. A bidder's sensations of gains and losses relative to the reference bid $\hat{b}$ seen from p depend on q. At any q, $p \leq q \leq b$, the bidder either wins or the auction goes on, thus a sensation of losing the good and gaining money is not possible. However, he wins with marginal probability $h(q|q)$ and the reference bid implies the probability $1 - H(\hat{b}|q)$ of not winning (which is 1 for all $\hat{b} < q$), given that q is reached, which happens with probability $1 - H(q|p)$. Because any pair of consequences from the decision and the reference point is evaluated independently according to (1), a gain of the good (and loss of money) is weighted by $(1-H(\hat{b}|q))h(q|q)(1-H(q|p)) = (1-H(\hat{b}|q))h(q|p)$ for any q with $p \leq q \leq b$ in our utility function. At any q, $p \leq b < q$ the bidder may not win the auction. Thus, a sensation of gaining the good and losing money is impossible. However, with the reference bid $\hat{b}$ his marginal probability of winning is $h(q|q)$, given that q is reached, which happens with probability $1 - H(q|p)$. Thus, a loss of the good and gain of money is weighted by $h(q|q)(1 - H(q|p)) = h(q|p)$ for any q with $p \leq b < q$.

A gain-loss averse bidder's utility function $U(A(p,b)|A(p,\hat{b}))$ if $b > \hat{b}$ is:

$$\begin{aligned}
U(A(p,b)|A(p,\hat{b})) &= \int_p^\infty \int_q^\infty u(q|s)h(s|q)h(q|p)\mathrm{d}s\mathrm{d}q \\
&= \int_p^b (v-q)h(q|p)\mathrm{d}q + \gamma_g v \int_p^{\hat{b}} h(q|p)(1-H(\hat{b}|q))\mathrm{d}q + \lambda_m \int_p^{\hat{b}} -qh(q|p)(1-H(\hat{b}|q))\mathrm{d}q \\
&\quad + \gamma_g v \int_{\hat{b}}^b h(q|p)\mathrm{d}q - \lambda_m \int_{\hat{b}}^b qh(q|p)\mathrm{d}q \\
&= \int_p^{\hat{b}} \Big(v(1+\gamma_g(1-H(\hat{b}|q))) - q(1+\lambda_m(1-H(\hat{b}|q)))\Big) h(q|p)\mathrm{d}q \\
&\quad + \int_{\hat{b}}^b \left(v(1+\gamma_g) - q(1+\lambda_m)\right) h(q|p)\mathrm{d}q
\end{aligned}$$

Figure A.3 visualizes the relevant cases for the derivation of the utility function.

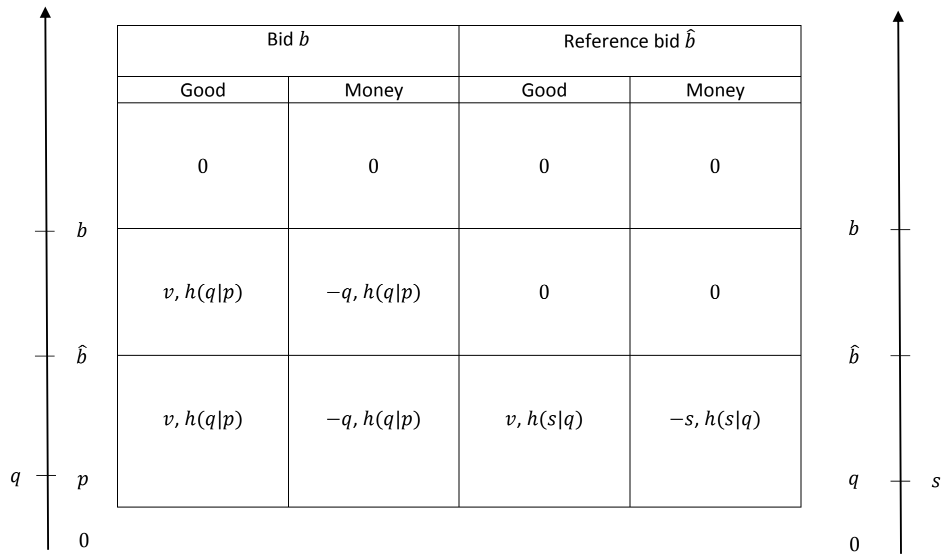

Figure A.3: The cell entries have the same meaning as those in Figure A.2.

Appendix A.2. Dutch Auction

In the DA, a gain-loss averse bidder's utility function $U(W(p,b)|W(p,\hat{b}))$ if $b \geq \hat{b}$ is:

$$
\begin{aligned}
U(W(p,b)|W(p,\hat{b})) &= \int_0^p \int_0^q u(q|s)g(s|q)g(q|p)\mathrm{d}s\mathrm{d}q \\
&= (v-b)G(b|p) + \lambda_g(0-v)\int_b^p g(q|p)G(\hat{b}|q)\mathrm{d}q + \gamma_m\hat{b}\int_b^p g(q|p)G(\hat{b}|q)\mathrm{d}q \\
&\quad + \gamma_g vG(b|p)\int_{\hat{b}}^b g(s|b)\mathrm{d}s + \lambda_m\left((-b)G(b|p)\int_{\hat{b}}^b g(s|b)\mathrm{d}s + (-b+\hat{b})G(b|p)G(\hat{b}|b)\right) \\
&= (v-b)G(b|p) - \lambda_g vG(\hat{b}|p)\int_b^p g(q|q)\mathrm{d}q + \gamma_m\hat{b}G(\hat{b}|p)\int_b^p g(q|q)\mathrm{d}q \\
&\quad + \gamma_g v\left(G(b|p) - G(\hat{b}|p)\right) - \lambda_m b\left(G(b|p) - G(\hat{b}|p)\right) - \lambda_m(b-\hat{b})G(\hat{b}|p) \\
&= (v-b)G(b|p) - \lambda_g vG(\hat{b}|p)(-\ln(G(b|p))) + \gamma_m\hat{b}G(\hat{b}|p)(-\ln(G(b|p))) \\
&\quad + \gamma_g v\left(G(b|p) - G(\hat{b}|p)\right) - \lambda_m\left(bG(b|p) - \hat{b}G(\hat{b}|p)\right)
\end{aligned}
$$

Remember, $G(b|p) = \frac{G(b)}{G(p)}$ and $g(b|p) = \frac{g(b)}{G(p)}$. Figure A.4 visualizes the relevant cases for the derivation of the utility function.

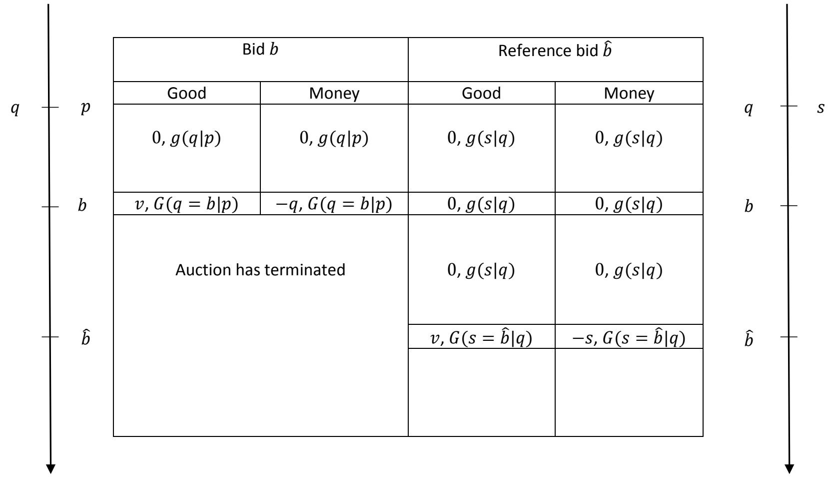

Figure A.4: 0 is written in a cell if at stage $q < p$ ($s < q$) the bid b (the reference bid $\hat{b}$) can turn out to be too low for being successful, because another bidder accepts the price q (s) and the auction ends at q (s). The other element in the cell is the marginal losing probability $g(q|p)$ ($g(s|q)$) at q (s), given that the auction has reached price level p (q), that is, the marginal probability with that not winning the auction is fixed. v or $-q$ ($-s$) is written in a cell that represents the single value $q = b$ ($s = \hat{b}$), at which the bid b (the reference bid $\hat{b}$) is successful, such that the auction ends at q (s) with the payment $q = b$ ($s = \hat{b}$). The other element in the cell is the winning probability, the probability with that $q = b$ ($s = \hat{b}$) is reached, seen from p (q). At levels $q < b$ the auction has terminated for sure. Levels $s < \hat{b}$ wouldn't be reached in an auction where the bid $\hat{b}$ is submitted. The variable q takes values smaller than p and the variable s takes values smaller than q.

A gain-loss averse gain-loss averse bidder's utility function $U(W(p,b)|W(p,\hat{b}))$ if $b < \hat{b}$ is:

$$\begin{aligned}
U(W(p,b)|W(p,\hat{b})) &= \int_0^p \int_0^q u(q|s)g(s|q)g(q|p)\mathrm{d}s\mathrm{d}q \\
&= (v-b)G(b|p) + \lambda_g(0-v)\int_{\hat{b}}^p g(q|p)G(\hat{b}|q)\mathrm{d}q + \gamma_m\hat{b}\int_b^p g(q|p)G(\hat{b}|q)\mathrm{d}q \\
&\quad + \lambda_g(0-v)\int_b^{\hat{b}} g(q|p)\mathrm{d}q + \gamma_m\hat{b}\int_b^{\hat{b}} g(q|p)\mathrm{d}q + \gamma_m(-b+\hat{b})G(b|p) \\
&= (v-b)G(b|p) - \lambda_g vG(\hat{b}|p)\int_{\hat{b}}^p g(q|q)\mathrm{d}q + \gamma_m\hat{b}G(\hat{b}|p)\int_b^p g(q|q)\mathrm{d}q \\
&\quad - \lambda_g v(G(\hat{b}|p) - G(b|p)) + \gamma_m\hat{b}(G(\hat{b}|p) - G(b|p)) + \gamma_m(-b+\hat{b})G(b|p) \\
&= (v-b)G(b|p) - \lambda_g vG(\hat{b}|p)\Big(-\ln(G(\hat{b}|p)) + 1 - G(b|\hat{b})\Big) \\
&\quad + \gamma_m G(\hat{b}|p)(-\hat{b}\ln(G(\hat{b}|p)) + \hat{b} - bG(b|\hat{b}))
\end{aligned}$$

Note that $G(s|q) = 1$ for $q \leq s$. Figure A.5 visualizes the relevant cases for the derivation of the utility function.

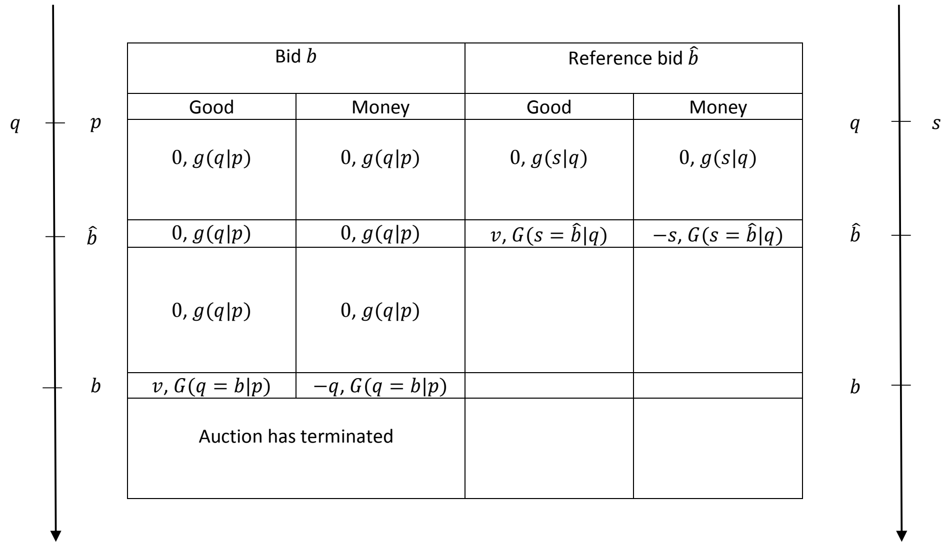

Figure A.5: The cell entries have the same meaning as those in Figure A.4.

Appendix B. Deriving the Continuous Approach from the Discrete Approach

Appendix B.1. English Auction

Let $t = 1, 2, 3, \ldots$ denote the bidding periods in an EA and $A(t, T)$ the lottery induced by the decision to bid from the present period t with price p_t up to period T with price p_T. A rational bidder who anticipates the future evaluates his decision in period t to stay in the auction until T taking all corresponding future lotteries $A(t+1, T), A(t+2, T), A(t+3, T), \ldots, A(T|T)$ into account. In the following, we derive the expected utility $U(A(t,T)|A(t,\widehat{T}))$ from decision $A(t,T)$ under the reference point $A(t,\widehat{T})$. Note, decision and reference point may differ.

Let $H(t)$ denote the probability that the bidder (who bids at least until t) will win the auction at price p_t in t.[19] Hence, when the auction is in $t' \geq 0$ (and has not ended in t'), for a bidder with bidding limit p_T $(T > t')$, the conditional probability of winning the auction in t $(T \geq t > t')$ and the complementary probability of not winning at t are given by

$$H(t|t') = \frac{H(t) - H(t')}{1 - H(t')} \qquad \text{and} \qquad 1 - H(t|t') = \frac{1 - H(t)}{1 - H(t')} \,. \tag{B.1}$$

[19] We ignore the possibility of a tie, because this analysis is mainly a means to derive the functions for the continuous case, in which such considerations do not play a role.

It is $H(t|t') = 0$ for $t' \geq t$. With $H(0) = 0$ it is $H(t|0) = H(t)$ for all $t \geq 1$.

All bidders $i \in N$ accept at least p_0. As an example for the calculations of probabilities consider period $t = 1$ when all bidders are asked to accept p_1 and a bidder with limit p_3 (i.e. in $t = 3$ he will bid for the last time). His probability of winning the auction in $t = 1$ at p_1 is $H(1)$, and his probability (viewed from period $t = 0$) of not winning the auction in $t = 1$ but winning the auction in $t = 2$ at p_2 is

$$(1 - H(1))H(2|1) = H(2) - H(1),$$

her probability of winning the auction in $t = 3$ at p_3 and not winning the auction in $t = 2$ and $t = 1$ is

$$(1 - H(1))(1 - H(2|1))H(3|2) = H(3) - H(2),$$

and his conditional probability of winning the auction in $t = 3$ at p_3 and not winning the auction in $t = 2$ after not winning the auction in $t = 1$ is

$$(1 - H(2|1))H(3|2) = \frac{H(3) - H(2)}{1 - H(1)}.$$

The expected utility $U(A(t,T)|A(t,\widehat{T}))$ for $t \in \{1, 2, \ldots, T\}$ is determined recursively. In period $t < \min\{T, \widehat{T}\}$ it is

$$\begin{aligned} U(A(t,T)|A(t,\widehat{T})) = H(t|t-1)\,&[v - p_t + (1 - H(\widehat{T}|t-1))(\gamma_g v - \lambda_m p_t)] \\ &+ (1 - H(t|t-1))\,U(A(t+1,T)|A(t+1,\widehat{T}))\,. \end{aligned} \tag{B.2}$$

For $T \leq \widehat{T}$, the expected utility in the last period $t = T$ is

$$\begin{aligned} U(A(T,T)|A(T,\widehat{T})) = H(T|T-1)\,&[v - p_T + (1 - H(\widehat{T}|T-1))(\gamma_g v - \lambda_m p_T)] \\ &+ (1 - H(T|T-1))\,[\gamma_m \textstyle\sum_{t'=T}^{\widehat{T}} (H(t'|T-1) - H(t'-1|T-1))\,p_{t'} \\ &- \lambda_g v\, H(\widehat{T}|T-1)] \end{aligned} \tag{B.3}$$

For $T > \widehat{T}$, the expected utility in period t with $T > t > \widehat{T}$ is

$$\begin{aligned} U(A(t,T)|A(t,\widehat{T})) = {} & H(t|t-1)\left[v - p_t + \gamma_g v - \lambda_m p_t\right] \\ & + (1 - H(t|t-1))\, U(A(t+1,T)|A(t+1,\widehat{T})) \end{aligned} \tag{B.4}$$

while in period $t = \widehat{T}$ it is

$$U(A(\widehat{T},T)|A(\widehat{T},\widehat{T})) = H(\widehat{T}|\widehat{T}-1)\left[v - p_t\right] + (1 - H(\widehat{T}|\widehat{T}-1))\, U(A(\widehat{T}+1,T)|A(\widehat{T}+1,\widehat{T})) \tag{B.5}$$

and in the last period $t = T$

$$U(A(T,T)|A(T,\widehat{T})) = H(T|T-1)\left[v - p_T + \gamma_g v - \lambda_m p_T\right]. \tag{B.6}$$

To compute the expected utility for $t < T \le \widehat{T}$ recursively, combine (B.2) and (B.3) with (B.1) to

$$
\begin{aligned}
U(A(t,T)|A(t,\widehat{T})) &= H(t|t-1)\left[v - p_t + (1 - H(\widehat{T}|t-1))(\gamma_g v - \lambda_m p_t)\right] \\
&\quad + (1 - H(t|t-1)) \cdot H(t+1|t) \cdot \left[v - p_{t+1} + (1 - H(\widehat{T}|t))(\gamma_g v - \lambda_m p_{t+1})\right] \\
&\quad + (1 - H(t|t-1)) \cdot (1 - H(t+1|t)) \cdot H(t+2|t+1) \cdot \left[v - p_{t+2} + (1 - H(\widehat{T}|t+1))(\gamma_g v - \lambda_m p_{t+2})\right] \\
&\quad + \dots \\
&\quad + (1 - H(t|t-1)) \cdot \ldots \cdot (1 - H(T-2|T-3)) \cdot H(T-1|T-2) \\
&\qquad \cdot \left[v - p_{T-1} + (1 - H(\widehat{T}|T-2))(\gamma_g v - \lambda_m p_{T-1})\right] \\
&\quad + (1 - H(t|t-1)) \cdot \ldots \cdot (1 - H(T-1|T-2)) \cdot H(T|T-1) \\
&\qquad \cdot \left[v - p_T + (1 - H(\widehat{T}|T-1))(\gamma_g v - \lambda_m p_T)\right] \\
&\quad + (1 - H(t|t-1)) \cdot \ldots \cdot (1 - H(T-1|T-2)) \cdot (1 - H(T|T-1)) \\
&\qquad \cdot \left[\gamma_m \textstyle\sum_{t'=T}^{\widehat{T}} (H(t'|T-1) - H(t'-1|T-1))\, p_{t'} - \lambda_g v\, H(\widehat{T}|T-1)\right] \\
&= \frac{1}{1 - H(t-1)} \Big(\left[v - p_t + (1 - H(\widehat{T}|t-1))(\gamma_g v - \lambda_m p_t)\right] \cdot (H(t) - H(t-1)) \\
&\quad + \left[v - p_{t+1} + (1 - H(\widehat{T}|t))(\gamma_g v - \lambda_m p_{t+1})\right] \cdot (H(t+1) - H(t)) \\
&\quad + \left[v - p_{t+2} + (1 - H(\widehat{T}|t+1))(\gamma_g v - \lambda_m p_{t+2})\right] \cdot (H(t+2) - H(t+1)) \\
&\quad + \dots \\
&\quad + \left[v - p_{T-1} + (1 - H(\widehat{T}|T-2))(\gamma_g v - \lambda_m p_{T-1})\right] \cdot (H(T-1) - H(T-2)) \\
&\quad + \left[v - p_T + (1 - H(\widehat{T}|T-1))(\gamma_g v - \lambda_m p_T)\right] \cdot (H(T) - H(T-1)) \\
&\quad + \gamma_m \textstyle\sum_{t'=T}^{\widehat{T}} ((H(t'|T-1) - H(t'-1|T-1))\, p_{t'} \cdot (1 - H(T))) - \lambda_g v\, (H(\widehat{T}|T-1) \cdot (1 - H(T))) \Big) \\
&= \sum_{i=t}^{T} (v - p_i) \frac{H(i) - H(i-1)}{1 - H(t-1)} + \sum_{i=t}^{T} (1 - H(\widehat{T}|i-1))(\gamma_g v - \lambda_m p_i) \frac{H(i) - H(i-1)}{1 - H(t-1)} \\
&\quad + \gamma_m \frac{1 - H(T)}{1 - H(T-1)} \sum_{t'=T}^{\widehat{T}} p_{t'} \cdot \frac{H(t') - H(t'-1)}{1 - H(t-1)} - \lambda_g v\, \frac{H(\widehat{T}) - H(T-1)}{1 - H(t-1)} \, \frac{1 - H(T)}{1 - H(T-1)} \qquad \text{(B.7)}
\end{aligned}
$$

To compute the expected utility for $t < \widehat{T} < T$ recursively, combine (B.2), (B.4), (B.5), and (B.6) with (B.1) to

$$
\begin{aligned}
U(A(t,T)|A(t,\widehat{T})) &= H(t|t-1)\,[v - p_t + (1 - H(\widehat{T}|t-1))(\gamma_g v - \lambda_m p_t)] \\
&+ (1 - H(t|t-1)) \cdot H(t+1|t) \cdot [v - p_{t+1} + (1 - H(\widehat{T}|t))(\gamma_g v - \lambda_m p_{t+1})] \\
&+ (1 - H(t|t-1)) \cdot (1 - H(t+1|t)) \cdot H(t+2|t+1) \cdot [v - p_{t+2} + (1 - H(\widehat{T}|t+1))(\gamma_g v - \lambda_m p_{t+2})] \\
&+ \dots \\
&+ (1 - H(t|t-1)) \cdot \ldots \cdot (1 - H(\widehat{T}-2|\widehat{T}-3)) \cdot H(\widehat{T}-1|\widehat{T}-2) \\
&\quad \cdot [v - p_{\widehat{T}-1} + (1 - H(\widehat{T}|\widehat{T}-2))(\gamma_g v - \lambda_m p_{\widehat{T}-1})] \\
&+ (1 - H(t|t-1)) \cdot \ldots \cdot (1 - H(\widehat{T}-1|\widehat{T}-2)) \cdot H(\widehat{T}|\widehat{T}-1)\,[v - p_{\widehat{T}}] \\
&+ (1 - H(t|t-1)) \cdot \ldots \cdot (1 - H(\widehat{T}|\widehat{T}-1)) \cdot H(\widehat{T}+1|\widehat{T}) \cdot [v - p_{\widehat{T}+1} + \gamma_g v - \lambda_m p_{\widehat{T}+1}] \\
&+ \dots \\
&+ (1 - H(t|t-1)) \cdot \ldots \cdot (1 - H(T-1|T-2)) \cdot H(T|T-1) \cdot [v - p_T + \gamma_g v - \lambda_m p_T] \\
&= \frac{1}{1 - H(t-1)} \Big([v - p_t + (1 - H(\widehat{T}|t-1))(\gamma_g v - \lambda_m p_t)] \cdot (H(t) - H(t-1)) \\
&+ [v - p_{t+1} + (1 - H(\widehat{T}|t))(\gamma_g v - \lambda_m p_{t+1})] \cdot (H(t+1) - H(t)) \\
&+ [v - p_{t+2} + (1 - H(\widehat{T}|t+1))(\gamma_g v - \lambda_m p_{t+2})] \cdot (H(t+2) - H(t+1)) \\
&+ \dots \\
&+ [v - p_{\widehat{T}-1} + (1 - H(\widehat{T}|\widehat{T}-2))(\gamma_g v - \lambda_m p_{\widehat{T}-1})] \cdot (H(\widehat{T}-1) - H(\widehat{T}-2)) \\
&+ [v - p_{\widehat{T}}] \cdot (H(\widehat{T}) - H(\widehat{T}-1)) \\
&+ [v - p_{\widehat{T}+1} + \gamma_g v - \lambda_m p_{\widehat{T}+1}] \cdot (H(\widehat{T}+1) - H(\widehat{T})) \\
&+ \dots \\
&+ [v - p_T + \gamma_g v - \lambda_m p_T] \cdot (H(T) - H(T-1)) \Big) \\
&= \sum_{i=t}^{T} (v - p_i) \frac{H(i) - H(i-1)}{1 - H(t-1)} + \sum_{i=t}^{\widehat{T}-1} (1 - H(\widehat{T}|i-1))(\gamma_g v - \lambda_m p_i) \frac{H(i) - H(i-1)}{1 - H(t-1)} \\
&+ \gamma_g v \frac{H(T) - H(\widehat{T})}{1 - H(t-1)} - \sum_{i=\widehat{T}+1}^{T} \lambda_m p_i \frac{H(i) - H(i-1)}{1 - H(t-1)} \qquad \text{(B.8)}
\end{aligned}
$$

Now, let p_t denote both the price level in period t and the period t itself and set $p := p_t$, $b := p_T$ with $p \leq b$, and $\hat{b} := p_{\widehat{T}}$. Let $\Delta p := p_t - p_{t-1}$ for all $t \geq 1$ and consider $\Delta p \to 0$. For $b \leq \hat{b}$ (i.e. $T \leq \widehat{T}$) we get from (B.7)

$$\begin{aligned}
U(A(p,b)|(A(p,\hat{b})) &= \lim_{\Delta p \to 0} U(A(t,T)|A(t,\widehat{T})) \\
&= \int_p^b v - s \,\mathrm{d}H(s|p) + \int_p^b (\gamma_g v - \lambda_m s)\left(1 - H(\hat{b}|s)\right)\,\mathrm{d}H(s|p) \\
&\quad + \gamma_m \int_b^{\hat{b}} s\ \mathrm{d}H(s|p) - \lambda_g v\,(H(\hat{b}|p) - H(b|p)) \\
&= \int_p^b v\left(1 + \gamma_g(1 - H(\hat{b}|s))\right) - s\left(1 + \lambda_m(1 - H(\hat{b}|s))\right)\,\mathrm{d}H(s|p) \\
&\quad + \int_b^{\hat{b}} \gamma_m s - \lambda_g v \,\mathrm{d}H(s|p),
\end{aligned} \tag{B.9}$$

and for $b > \hat{b}$ (i.e. $T > \widehat{T}$) we get from (B.8)

$$\begin{aligned}
U(A(p,b)|(A(p,\hat{b})) &= \lim_{\Delta p \to 0} U(A(t,T)|A(t,\widehat{T})) \\
&= \int_p^b v - s\,\mathrm{d}H(s|p) + \gamma_g v\left(\int_p^{\hat{b}} 1 - H(\hat{b}|s)\,\mathrm{d}H(s|p) + H(b|p) - H(\hat{b}|p)\right) \\
&\quad - \lambda_m\left(\int_p^{\hat{b}} s\,(1 - H(\hat{b}|s))\,\mathrm{d}H(s|p) + \int_{\hat{b}}^b s\,\mathrm{d}H(s|p)\right) \\
&= \int_p^{\hat{b}} v\left(1 + \gamma_g(1 - H(\hat{b}|s))\right) - s\left(1 + \lambda_m(1 - H(\hat{b}|s))\right)\,\mathrm{d}H(s|p) \\
&\quad + \int_{\hat{b}}^b v(1 + \gamma_g) - s(1 + \lambda_m)\,\mathrm{d}H(s|p).
\end{aligned} \tag{B.10}$$

Appendix B.2. Dutch Auction

Let $t = 1, 2, 3, \ldots$ denote the periods in a DA and $W(t,T)$ the lottery induced by the decision to wait from the present period t with price p_t until period T with price p_T and to bid in T. In a Dutch auction price levels decrease over time such that $p_t > p_{t+1}$. A rational bidder who anticipates the future evaluates his decision in period t to wait with bidding until T taking all corresponding future lotteries $W(t+1,T), W(t+2,T), W(t+3,T), \ldots, W(T|T)$ into account. In the following, we derive the expected utility $U(W(t,T)|W(t,\widehat{T}))$ from decision $W(t,T)$ under the reference point $W(t, \widehat{T})$. Note, decision and reference point may differ.

Let $G(t)$ denote the probability that the highest bid is below p_t, that is, that the auction does not end before t. Thus, $G(T)$ is the probability that the bidder who waits until T will win the auction (at price p_T in T).[20] Hence, when the auction is in $t' < T$ (and bids at t' have not yet been evaluated), for a bidder with bidding limit p_T, the conditional probability of winning the auction in T and the complementary probability of not winning at T are given by

$$G(T|t') = \frac{G(T)}{G(t')} \qquad \text{and} \qquad 1 - G(T|t') = \frac{G(t') - G(T)}{G(t')}\,. \tag{B.11}$$

It is $G(t|t') = \frac{G(t)}{G(t')}$ for $t' < t < T$ but the probability to win in t for a bidder who bids in T is zero – either he wins in T or someone else wins at $t < T$. With $G(0) = 1$ it is $G(t|0) = G(t)$ for all $t \geq 1$.

Bidders $i \in N$ participate. As an example for the calculations of probabilities consider period $t = 0$ when all bidders are asked whether they bid p_0, and a bidder with limit p_3 (i.e. in $t = 3$ he will bid). He may either win the auction at p_3 or not win the auction at all. His probability of winning the auction in $T = 3$ at p_3 is $G(3)$, his probability of winning the auction in $T = 3$ when $t = 0$ has been evaluated and $t = 1$ is running (viewed from $t = 0$) is

$$G(1)G(3|1) = G(3)$$

her probability of winning the auction in $T = 3$ at p_3 when $t = 1$ has been evaluated and $t = 2$ is running (viewed from $t = 0$ or $t = 1$) is

$$G(2)G(3|2) = G(3),$$

and his conditional probability of winning the auction in $T = 3$ at p_3 and when $t = 2$ has been

[20] Again, we ignore the possibility of a tie, because this analysis is mainly a means to derive the functions for the continuous case, in which such considerations do not play a role.

evaluated and $t = 3$ is running (viewed from $t \leq 2$) is

$$G(3|2) = \frac{G(3)}{G(2)}.$$

From bidder i's point of view, the probability that someone else wins the item in t is $(1-G(t+1|t))$, because $G(t+1|t)$ is the probability that $t+1$ is reached when t has not yet been evaluated.

The expected utility $U(W(t,T)|W(t,\widehat{T}))$ for $t \in \{1, 2, \ldots, T\}$ is determined recursively. In period $t < \min\{T-1, \widehat{T}\}$ is

$$\begin{aligned} U(W(t,T)|W(t,\widehat{T})) = (1 - G(t+1|t))G(\widehat{T}|t)\,[-\lambda_g v + \gamma_m p_{\widehat{T}}] \\ + G(t+1|t)\,U(W(t+1,T)|W(t+1,\widehat{T}))\,. \end{aligned} \tag{B.12}$$

For $T < \widehat{T}$, the expected utility in the last period $t = T$ is

$$\begin{aligned} U(W(T,T)|W(T,\widehat{T})) &= v - p_T + (1 - G(\widehat{T}|T))[\gamma_g v - \lambda_m p_T] - G(\widehat{T}|T)\lambda_m(p_T - p_{\widehat{T}}) \\ &= v - p_T + (1 - G(\widehat{T}|T))\gamma_g v - \lambda_m(p_T - G(\widehat{T}|T)p_{\widehat{T}}) \\ &= v - p_T + (1 - G(\widehat{T}|T))\gamma_g v - \lambda_m(G(T|T)p_T - G(\widehat{T}|T)p_{\widehat{T}}), \end{aligned} \tag{B.13}$$

for $T = \widehat{T}$, the expected utility in the last period $t = T$ is

$$U(W(T,T)|W(T,T)) = v - p_T, \tag{B.14}$$

For $T > t \geq \widehat{T}$ the expected utility is

$$U(W(t,T)|W(t,\widehat{T})) = (1 - G(t+1|t))\,[-\lambda_g v + \gamma_m p_{\widehat{T}}]. \tag{B.15}$$

and in the last period $t = T$ it is

$$U(W(T,T)|W(T,\widehat{T})) = v - p_T + \gamma_m(p_{\widehat{T}} - p_T). \tag{B.16}$$

To compute the expected utility for $t < T < \widehat{T}$ recursively, combine (B.12) and (B.13) with (B.11) to

$$
\begin{aligned}
&U(W(t,T)|W(t,\widehat{T})) = \\
&\quad (1 - G(t+1|t))G(\widehat{T}|t)\,[-\lambda_g v + \gamma_m p_{\widehat{T}}] \\
&\quad + G(t+1|t)\,(1 - G(t+2|t+1))G(\widehat{T}|t+1)\,[-\lambda_g v + \gamma_m p_{\widehat{T}}] \\
&\quad + G(t+1|t)\,G(t+2|t+1)\,(1 - G(t+3|t+2))G(\widehat{T}|t+2)\,[-\lambda_g v + \gamma_m p_{\widehat{T}}] \\
&\quad + \dots \\
&\quad + G(t+1|t)\cdot \ldots \cdot G(T-1|T-2)(1 - G(T|T-1))G(\widehat{T}|T-1)\,[-\lambda_g v + \gamma_m p_{\widehat{T}}] \\
&\quad + G(t+1|t)\cdot \ldots \cdot G(T-1|T-2)G(T|T-1)\,[v - p_T + (1 - G(\widehat{T}|T))\gamma_g v - \lambda_m(p_T - G(\widehat{T}|T)p_{\widehat{T}})] \\
&= [-\lambda_g v + \gamma_m p_{\widehat{T}}]\,(G(\widehat{T})/G(t)\sum_{i=t}^{T-1}(G(i) - G(i+1))/G(i)) \\
&\quad + [v - p_T + \gamma_g v(1 - G(\widehat{T}|T)) - \lambda_m(p_T - G(\widehat{T}|T)p_{\widehat{T}})]\,G(T)/G(t) \\
&= [v - p_T]\,G(T|t) + [-\lambda_g v + \gamma_m p_{\widehat{T}}]\,G(\widehat{T}|t)\sum_{i=t}^{T-1}(G(i) - G(i+1))/G(i)) \\
&\quad + \gamma_g v(G(T|t) - G(\widehat{T}|t)) - \lambda_m(p_T G(T|t) - p_{\widehat{T}} G(\widehat{T}|t)).
\end{aligned}
\tag{B.17}
$$

To compute the expected utility for $t < T = \widehat{T}$ recursively, combine (B.12) and (B.14) with (B.11) to

$$
\begin{aligned}
&U(W(t,T)|W(t,T)) = \\
&\quad (1 - G(t+1|t))G(T|t)\,[-\lambda_g v + \gamma_m p_T] \\
&\quad + G(t+1|t)\,(1 - G(t+2|t+1))G(T|t+1)\,[-\lambda_g v + \gamma_m p_T] \\
&\quad + G(t+1|t)\,G(t+2|t+1)\,(1 - G(t+3|t+2))G(T|t+2)\,[-\lambda_g v + \gamma_m p_T] \\
&\quad + \ldots \\
&\quad + G(t+1|t) \cdot \ldots \cdot G(T-1|T-2)(1 - G(T|T-1))G(T|T-1)\,[-\lambda_g v + \gamma_m p_T] \\
&\quad + G(t+1|t) \cdot \ldots \cdot G(T-1|T-2)G(T|T-1)\,[v - p_T] \\
&= [-\lambda_g v + \gamma_m p_T]\,(G(T)/G(t) \sum_{i=t}^{T-1} (G(i) - G(i+1))/G(i)) + [v - p_T]\,G(T)/G(t) \\
&= [v - p_T]\,G(T|t) + [-\lambda_g v + \gamma_m p_T]\,(G(T|t) \sum_{i=t}^{T-1} (G(i) - G(i+1))/G(i)).
\end{aligned}
\tag{B.18}
$$

To compute the expected utility for $t < \widehat{T} < T$ recursively, combine (B.12), (B.15), and (B.16) with (B.11) to

$$
\begin{aligned}
&U(W(t,T)|W(t,\widehat{T})) = \\
&\quad (1 - G(t+1|t))G(\widehat{T}|t)\,[-\lambda_g v + \gamma_m p_{\widehat{T}}] \\
&\quad + G(t+1|t)\,(1 - G(t+2|t+1))G(\widehat{T}|t+1)\,[-\lambda_g v + \gamma_m p_{\widehat{T}}] \\
&\quad + G(t+1|t)\,G(t+2|t+1)\,(1 - G(t+3|t+2))G(\widehat{T}|t+2)\,[-\lambda_g v + \gamma_m p_{\widehat{T}}] \\
&\quad + \dots \\
&\quad + G(t+1|t)\cdot \ldots \cdot G(\widehat{T}-1|\widehat{T}-2)(1 - G(\widehat{T}|\widehat{T}-1))G(\widehat{T}|\widehat{T}-1)\,[-\lambda_g v + \gamma_m p_{\widehat{T}}] \\
&\quad + G(t+1|t)\cdot \ldots \cdot G(\widehat{T}|\widehat{T}-1)(1 - G(\widehat{T}+1|\widehat{T}))\,[-\lambda_g v + \gamma_m p_{\widehat{T}}] \\
&\quad + G(t+1|t)\cdot \ldots \cdot G(\widehat{T}+1|\widehat{T})(1 - G(\widehat{T}+2|\widehat{T}+1))\,[-\lambda_g v + \gamma_m p_{\widehat{T}}] \\
&\quad + \dots \\
&\quad + G(t+1|t)\cdot \ldots \cdot G(T-1|T-2)(1 - G(T|T-1))\,[-\lambda_g v + \gamma_m p_{\widehat{T}}] \\
&\quad + G(t+1|t)\cdot \ldots \cdot G(T|T-1)\,[v - p_T + \gamma_m(p_{\widehat{T}} - p_T)] \\
&= [-\lambda_g v + \gamma_m p_{\widehat{T}}]\,[(G(\widehat{T})/G(t)\sum_{i=t}^{\widehat{T}-1}(G(i) - G(i+1))/G(i)) + (G(\widehat{T}) - G(T))/G(t)] \\
&\quad + [v - p_T + \gamma_m(p_{\widehat{T}} - p_T)]\,G(T)/G(t) \\
&= [-\lambda_g v + \gamma_m p_{\widehat{T}}]\,[(G(\widehat{T}|t)\sum_{i=t}^{\widehat{T}-1}(G(i) - G(i+1))/G(i)) + G(\widehat{T}|t) - G(T|t)] \\
&\quad + [v - p_T + \gamma_m(p_{\widehat{T}} - p_T)]\,G(T|t) \\
&= [v - p_T]\,G(T|t) - \lambda_g v\,G(\widehat{T}|t)[1 - G(T|\widehat{T}) - \sum_{i=t}^{\widehat{T}-1}(G(i+1) - G(i))/G(i))] \\
&\quad + \gamma_m\,G(\widehat{T}|t)[p_{\widehat{T}}(1 - \sum_{i=t}^{\widehat{T}-1}(G(i+1) - G(i))/G(i))) - p_T G(T|\widehat{T})]. \qquad \text{(B.19)}
\end{aligned}
$$

Now, let p_t denote the price level in period t and also refer by p_t to the period t itself (but note that higher prices come with earlier periods) and set $p := p_t$, $b := p_T$, and $\hat{b} := p_{\widehat{T}}$, with $p \geq \max\{b, \hat{b}\}$. Let $\Delta p := p_{t-1} - p_t$ for all $t \geq 1$ and consider $\Delta p \to 0$.

For $b \geq \hat{b}$ (i.e. $T \leq \widehat{T}$) we get from (B.17) and (B.18)

$$\begin{aligned}
U(W(p,b)|W(p,\hat{b})) &= (v-b)\,G(b|p) - (\lambda_g v - \gamma_m \hat{b})\,G(\hat{b}|p) \int_b^p g(s)/G(s)\,\mathrm{d}s \\
&\qquad + \gamma_g v\,(G(b|p) - G(\hat{b}|p)) - \lambda_m\,(b\,G(b|p) - \hat{b}\,G(\hat{b}|p)) \\
&= (v-b)\,G(b|p) + (\lambda_g v - \gamma_m \hat{b})\,G(\hat{b}|p) \ln(G(b|p)) \\
&\qquad + \gamma_g v\,(G(b|p) - G(\hat{b}|p)) - \lambda_m\,(b\,G(b|p) - \hat{b}\,G(\hat{b}|p))
\end{aligned} \tag{B.20}$$

and for $b < \hat{b}$ (i.e. $T > \widehat{T}$) we get from (B.19)

$$\begin{aligned}
U(W(p,b)|W(p,\hat{b})) &= (v-b)\,G(b|p) - \lambda_g v\,G(\hat{b}|p)\,[1 - G(b|\hat{b}) - \int_p^{\hat{b}} g(s)/G(s)\,\mathrm{d}s] \\
&\qquad + \gamma_m\,G(\hat{b}|p)\,[\hat{b}(1 - \int_p^{\hat{b}} g(s)/G(s)\,\mathrm{d}s) - bG(b|\hat{b})] \\
&= (v-b)\,G(b|p) - \lambda_g v\,G(\hat{b}|p)\,[1 - G(b|\hat{b}) - \ln(G(\hat{b}|p))] \\
&\qquad + \gamma_m\,G(\hat{b}|p)\,[\hat{b}(1 - \ln(G(\hat{b}|p))) - bG(b|\hat{b})].
\end{aligned} \tag{B.21}$$

Appendix C. Proofs for the English Auction

Proposition 1. *Given a value v and the beliefs $H(\cdot)$, b^* is a PE in the EA if and only if*

(EA1) $U(A(p,b^*)|A(p,b^*)) \geq U(A(p,p)|A(p,b^*))$ *for all* $p \leq b^*$ *and*

(EA2) $U(A(b^*,b^*)|A(b^*,b^*)) \geq U(A(b^*,b)|A(b^*,b^*))$ *for all* $b > b^*$.

Proof: Using the definition of a PE (Definition 1), we show that $U(A(p,b^*)|A(p,b^*)) \geq U(A(p,b)|A(p,b^*))$ for all b and $p \leq \min\{b^*, b\}$ if and only if conditions (EA1) and (EA2) are fulfilled.
if: Express $U(A(p,b)|A(p,b^*))$ for $p \leq \min\{b, b^*\}$ as

$$\begin{aligned}
&U(A(p,b^*)|A(p,b^*)) \\
&= \begin{cases} U(A(p,b)|A(p,b^*)) + (1 - H(b|p))\big(U(A(b,b^*)|A(b,b^*)) - U(A(b,b)|A(b,b^*))\big) & \text{if } b \leq b^* \\ U(A(p,b)|A(p,b^*)) - (1 - H(b^*|p))U(A(b^*,b)|A(b^*,b^*)) & \text{if } b > b^* \end{cases}
\end{aligned}$$

using (4), (6), (5) for $b \leq b^*$, and (4), (6), (8) for $b > b^*$, as well as (2), and replacing $\mathrm{d}H(s|x)$ by $h(s)/(1 - H(x))\mathrm{d}s$.

Note that $1 - H(b|p) \geq 0$ and $1 - H(b^*|p) \geq 0$. It follows that $U(A(p, b^*)|A(p, b^*)) \geq U(A(p, b)|A(p, b^*))$ for all b because by (EA1) it is $U(A(b, b^*)|A(b, b^*)) \geq U(A(b, b)|A(b, b^*)$ for $b \leq b^*$ and by (EA2) it is $U(A(b^*, b^*)|A(b^*, b^*)) = 0 \geq U(A(b^*, b)|A(b^*, b^*))$ for $b > b^*$.

only if:

For $p = b < b^*$, $U(A(p, b^*)|A(p, b^*)) \geq U(A(p, b)|A(p, b^*))$ is equal to (EA1).

For $p = b^* < b$, $U(A(p, b^*)|A(p, b^*)) \geq U(A(p, b)|A(p, b^*))$ is equal to (EA2). ■

Proposition 2. *For a bidder with the value v in the EA,*

(a) *a PE exists if and only if $\gamma_m(1 + \gamma_g) \leq \lambda_g(1 + \lambda_m)$,*

(b) *for every PE b^* it holds that*

$$b^* \in \left[v\frac{1 + \gamma_g}{1 + \lambda_m}, v\frac{1 + \lambda_g + \gamma_g}{1 + \lambda_m + \gamma_m} \right],$$

(c) *b^* is a PE for any beliefs $H(\cdot)$ if and only if $\gamma_m(1 + \gamma_g) \leq \lambda_g(1 + \lambda_m)$ and*

$$b^* \in \left[v\frac{1 + \gamma_g}{1 + \lambda_m}, v \min\left\{ \frac{1 + \lambda_g + \gamma_g}{1 + \lambda_m + \gamma_m}, \frac{1 + \lambda_g}{1 + \gamma_m} \right\} \right],$$

(d) *for any $H(\cdot)$, if b^* is a PE, every $b \in \left[v\frac{1+\gamma_g}{1+\lambda_m}, b^* \right)$ is also a PE.*

Proof: According to Proposition 1, b^* is a PE if it fulfills the two conditions (EA1) and (EA2). Using the utility functions (6) and (5), (EA1) becomes

$$\begin{aligned} & U(A(p, b^*)|A(p, b^*)) - U(A(p, p)|A(p, b^*)) \\ &= \int_p^{b^*} (v - s) + (\gamma_g v - \lambda_m s)(1 - H(b^*|s)) \, \mathrm{d}H(s|p) + \int_p^{b^*} \lambda_g v - \gamma_m s \, \mathrm{d}H(s|p) \\ &= \int_p^{b^*} v(1 + \lambda_g + \gamma_g(1 - H(b^*|s))) - s(1 + \lambda_m(1 - H(b^*|s)) + \gamma_m) \, \mathrm{d}H(s|p) \geq 0 \;\; \forall \, p \leq b^*. \end{aligned} \tag{C.1}$$

Using the utility functions (7) and (8), (EA2) becomes

$$U(A(b^*,b^*)|A(b^*,b^*)) - U(A(b^*,b)|A(b^*,b^*))$$
$$= \int_{b^*}^{b} s(1+\lambda_m) - v(1+\gamma_g)\, \mathrm{d}H(s|b^*) \geq 0 \;\; \forall\, b > b^*. \tag{C.2}$$

For $v = 0$, (C.1) is violated for all $b^* > 0$, and both (C.1) and (C.2) are fulfilled for $b^* = 0$. So, $\beta(0) = 0$, and in what follows we restrict attention to $v > 0$.

Proof of (a): We first prove that if a PE b^* exists, then $\lambda_g(1+\lambda_m) \geq \gamma_m(1+\gamma_g)$.
Condition (C.1) can only be fulfilled if $b^* \leq \bar{b} := v\frac{1+\lambda_g+\gamma_g}{1+\lambda_m+\gamma_m}$ since $H(b^*|s) = \frac{H(b^*)-H(s)}{1-H(s)} \to 0$ for $s \to b^*$ and thus $(1 - H(b^*|s)) \to 1$ for $s \to b^*$. If, to the contrary, $b^* > \bar{b}$, the integrand in Condition (C.1) is negative for $s > \bar{b}$ and $H(b^*|s) = 0$, and, thus, (C.1) is violated for p close to b^* for any $H(\cdot)$ with positive probability mass on $[p, b^*]$ (which holds by Assumption 1).
Condition (C.2) is fulfilled if and only if $b^* \geq \underline{b} := v\frac{1+\gamma_g}{1+\lambda_m}$. If $b^* \geq \underline{b}$, the integrand in Condition (C.2) is positive and the condition is fulfilled. If, to the contrary, $b^* < \underline{b}$, the integral in Condition (C.2) is negative for all b with $b^* < b \leq \underline{b}$ because the integrand is negative for all $b^* < s < \underline{b}$, and the condition is violated for any $H(\cdot|b^*)$ with positive probability mass on $[b^*, b]$ (which holds by Assumption 1).
Thus, for any PE b^* it holds that $\underline{b} \leq b^* \leq \bar{b}$, that is,

$$\underline{b} \leq \bar{b} \quad \iff \quad v\frac{1+\gamma_g}{1+\lambda_m} \leq v\frac{1+\lambda_g+\gamma_g}{1+\lambda_m+\gamma_m} \quad \iff \quad \gamma_m(1+\gamma_g) \leq \lambda_g(1+\lambda_m). \tag{C.3}$$

We now prove that if $\gamma_m(1+\gamma_g) \leq \lambda_g(1+\lambda_m)$ then a PE b^* exists. As shown above, $\gamma_m(1+\gamma_g) \leq \lambda_g(1+\lambda_m)$ if and only if $\underline{b} \leq \bar{b}$ (see (C.3)) and Condition (C.2) requires $b^* \geq \underline{b}$. It remains to show that there exists a $b \in [\underline{b}, \bar{b}]$ that fulfills Condition (C.1). By Proposition 1, this b must be a PE.
First, we show if $\gamma_g(1+\gamma_m) \leq \lambda_m(1+\lambda_g)$, Condition (C.1) is fulfilled for all $b \leq \bar{b}$, and any $b \in [\underline{b}, \bar{b}]$ is a PE. Replace $1 - H(b|s)$ by $\alpha \in (0,1]$ and consider the integrand in (C.1),

$$v(1+\lambda_g+\gamma_g\alpha) - s(1+\lambda_m\alpha+\gamma_m). \tag{C.4}$$

For $s \leq \bar{b}$, $v > 0$, and $\alpha < 1$ the integrand (C.4) is non-negative because for $s = \bar{b} = v\frac{1+\lambda_g+\gamma_g}{1+\lambda_m+\gamma_m}$

$$v(1+\lambda_g+\alpha\gamma_g) \geq v\frac{1+\lambda_g+\gamma_g}{1+\lambda_m+\gamma_m}(1+\lambda_m\alpha+\gamma_m) \iff (1-\alpha)\lambda_m(1+\lambda_g) \geq (1-\alpha)\gamma_g(1+\gamma_m).$$

Thus, Condition (C.1) is fulfilled for all $b \leq \bar{b}$. Since there is positive probability mass between $s < \bar{b}$ and $\bar{b}$, it is $\alpha < 1$ and (C.4) is positive, and Condition (C.1) is strictly fulfilled for all $p < b$ and $b \leq \bar{b}$.

Second, we show if $\gamma_g(1+\gamma_m) > \lambda_m(1+\lambda_g)$, Condition (C.1) is fulfilled for all $b \leq \hat{b} := v\frac{1+\lambda_g}{1+\gamma_m}$, and any $b \in [\underline{b}, \hat{b}]$ is a PE. Note that $\hat{b} < \bar{b}$ if and only if $\gamma_g(1+\gamma_m) > \lambda_m(1+\lambda_g)$ because

$$\hat{b} < \bar{b} \quad\iff\quad v\frac{1+\lambda_g}{1+\gamma_m} < v\frac{1+\lambda_g+\gamma_g}{1+\lambda_m+\gamma_m} \quad\iff\quad \gamma_g(1+\gamma_m) > \lambda_m(1+\lambda_g), \tag{C.5}$$

and that $\hat{b} \geq \underline{b}$ because $\lambda_g \geq \gamma_g$ and $\lambda_m \geq \gamma_m$. For $s \leq \hat{b}$ and $v > 0$, the integrand (C.4) is positive because for $s = \hat{b}$

$$v(1+\lambda_g+\alpha\gamma_g) > v\frac{1+\lambda_g}{1+\gamma_m}(1+\alpha\lambda_m+\gamma_m) \quad\iff\quad \alpha\gamma_g(1+\gamma_m) > \alpha\lambda_m(1+\lambda_g).$$

Since there is positive probability mass between $s < \hat{b}$ and $\hat{b}$, Condition (C.1) is strictly fulfilled for all $p < b$ and $b \leq \hat{b}$.

Proof of (b): In the proof of (a) we have already shown that for any PE b^* it holds that $\underline{b} \leq b^* \leq \bar{b}$.

Proof of (c): By (a) and (b) we know that $\gamma_m(1+\gamma_g) \leq \lambda_g(1+\lambda_m)$ is necessary and sufficient for existence and any PE must be in $\left[\underline{b}, \bar{b}\right]$. By the proof of (a), if $\gamma_m(1+\gamma_g) \leq \lambda_g(1+\lambda_m)$ then $\min\{\bar{b}, \hat{b}\} \geq \underline{b}$, Condition (C.2) is fulfilled for all candidates $b^* \geq \underline{b}$, and Condition (C.1) is fulfilled for all $b^* \in [\underline{b}, \min\{\bar{b}, \hat{b}\}]$ for any $H(\cdot)$ (if Assumption 1 applies). If $\hat{b} < \bar{b}$, there might be PE $b^* \in (\hat{b}, \bar{b}]$, because for $v > 0$, according to the proof of (a), the integrand (C.4) is strictly positive for all $s \leq \hat{b}$ and might be positive for $s > \hat{b}$. Thus, any potential PE $b^* \in (\hat{b}, \bar{b}]$ is belief-dependent. For beliefs with sufficiently high $H(s|b^*)$ at $s > \hat{b}$ (i.e., α sufficiently small) the integrand (C.4) might be negative for $p \leq s \leq b^*$ for p close to b^*, a contradiction. For example, for $s := \hat{b} + \varepsilon v/(1+\gamma_m) \leq \bar{b}$ the integrand (C.4) is negative if $\alpha(\gamma_g(1+\gamma_m) - \lambda_m(1+\lambda_g)) < \varepsilon(1+\alpha\lambda_m+\gamma_m)$, which holds for sufficiently small α and sufficiently large ε.

Proof of (d): By (c), part (d) holds true for any $H(\cdot)$ for any $b \in [\underline{b}, \min\{\bar{b}, \hat{b}\}]$, and by (b) any

PE is in $[\underline{b}, \bar{b}]$. It remains to show that for any $H(\cdot)$, if $\hat{b} < \bar{b}$ and $b^* \in [\hat{b}, \bar{b}]$ is a PE, every $b \in \left[v\frac{1+\gamma_g}{1+\lambda_m}, b^*\right)$ is also a PE. In the proof of (c) we argued that such a belief-dependent PE may exist. Assume that b^* is such a PE. For $b < b^*$ it is $H(b|s) \leq H(b^*|s)$ for all $s \in [p, b]$. The integrands in Condition (C.1) are never smaller for b than for b^*, because for $v > 0$ and $b < \bar{b}$

$$
\begin{aligned}
&v(1+\lambda_g+\gamma_g(1-H(b|s))) - s(1+\lambda_m(1-H(b|s))+\gamma_m) \\
&\qquad \geq v(1+\lambda_g+\gamma_g(1-H(b^*|s))) - s(1+\lambda_m(1-H(b^*|s))+\gamma_m) \quad \text{for all } s \in [p, b] \\
\iff &\qquad v\gamma_g(H(b^*|s)) - H(b|s)) \geq s\lambda_m(H(b^*|s)) - H(b|s)) \qquad \text{for all } s \in [p, b] \\
\iff &\qquad v\gamma_g \geq s\lambda_m \qquad \text{for all } s \in [p, b] \\
\Longleftarrow &\qquad v\gamma_g > \bar{b}\lambda_m = v\lambda_m \frac{1+\lambda_g+\gamma_g}{1+\lambda_m+\gamma_m} \\
\iff &\qquad \gamma_g(1+\lambda_m+\gamma_m) > \lambda_m(1+\lambda_g+\gamma_g) \\
\iff &\qquad \gamma_g(1+\gamma_m) > \lambda_m(1+\lambda_g) \\
\iff &\qquad \hat{b} < \bar{b},
\end{aligned}
$$

where the last equivalence holds by (C.5) and the last line is the condition of the considered case.

■

Proposition 3. *Given a value v and the beliefs $H(\cdot)$, if $\lambda_m = 0$, a PPE exists and $b^{**} = v(1+\gamma_g)$ is his unique PPE.*

Proof: If $\lambda_m = 0$, a PE exists without further restrictions because the condition for existence in Proposition 2(a) becomes $0 \leq \lambda_g$. Because b^{**} is the smallest PE, Condition (2a) of Definition 2 does not apply. Conditions (1) and (2b) hold, that is, for all PE $b > b^{**}$ for all $p \leq b^{**}$ it is $U(A(p, b^{**})|A(p, b^{**})) > U(A(p, b)|A(p, b))$ and for all $b^{**} < p \leq b$ it is $0 \geq U(A(p, b)|A(p, b))$. For all $p \leq b^{**}$,

$$
\begin{aligned}
&U(A(p, b^{**})|A(p, b^{**})) = \int_p^{b^{**}} v\big(1+\gamma_g\,(1-H(b^{**}|s))\big) - s\,\mathrm{d}H(s|p) \\
&\quad > \int_p^{b^{**}} v\big(1+\gamma_g\,(1-H(b|s))\big) - s\,\mathrm{d}H(s|p) + \int_{b^{**}}^{b} v\big(1+\gamma_g\,(1-H(b|s))\big) - s\,\mathrm{d}H(s|p) \\
&\quad = U(A(p, b)|A(p, b)),
\end{aligned}
$$

because $H(b^{**}|s) < H(b|s)$ and $v\big(1+\gamma_g\,(1-H(b|s))\,\big)-b^{**} < 0$, and, using the same relationships, for all $b^{**} < p \leq b$

$$U(A(b^{**}, b^{**})|A(b^{**}, b^{**})) = 0 \geq \int_p^b v\big(1 + \gamma_g\,(1 - H(b|s))\,\big) - s\,\mathrm{d}H(s|p) = U(A(p, b)|A(p, b)).$$

For any other PE, Condition (1) is violated. Thus, b^{**} is the unique PPE. ■

Proposition 10. *Consider bidders that assign the good and money the same consumption dimension, the "monetary rent dimension." A bidder's only PE and thus his unique PPE in the EA is $\beta^*(v) = v$. All bidders choosing $\beta^*(v) = v$ constitutes the unique PE profile and the unique PPE profile.*

Proof: Consider a bidder that assigns the good and money the same consumption dimension, the "monetary rent dimension" with parameters λ and γ. His utility $U(A(p, b)|A(p, b'))$ is, if (1) $b \leq b'$, and (a) $b' < v$, (b) $b \leq v \leq b'$, (c) $v < b$

$$\begin{cases} (a)\ \int_0^b (v-s)[1+\gamma(1-H(b'|s))]\mathrm{d}H(s|p) - \lambda \int_b^{b'} (v-s)\mathrm{d}H(s|p) \\ (b)\ \int_0^b (v-s)[1+\gamma(1-H(b'|s))]\mathrm{d}H(s|p) - \lambda \int_b^{v} (v-s)\mathrm{d}H(s|p) + \gamma \int_v^{b'} (s-v)\mathrm{d}H(s|p) \\ (c)\ \int_0^v (v-s)[1+\gamma(1-H(b'|s))]\mathrm{d}H(s|p) - \int_v^{b} (s-v)[1+\lambda(1-H(b'|s))]\mathrm{d}H(s|p) + \gamma \int_b^{b'} (s-v)\mathrm{d}H(s|p), \end{cases}$$

and if (2) $b > b'$, and (a) $b < v$, (b) $b' \leq v \leq b$, (c) $v < b'$

$$\begin{cases} (a)\ \int_0^{b'} (v-s)[1+\gamma(1-H(b'|s))]\mathrm{d}H(s|p) + \int_{b'}^{b} (v-s)(1+\gamma)\mathrm{d}H(s|p) \\ (b)\ \int_0^{b'} (v-s)[1+\gamma(1-H(b'|s))]\mathrm{d}H(s|p) + \int_{b'}^{v} (v-s)(1+\gamma)\mathrm{d}H(s|p) - \int_v^{b} (s-v)(1+\lambda)\mathrm{d}H(s|p) \\ (c)\ \int_0^{v} (v-s)[1+\gamma(1-H(b'|s))]\mathrm{d}H(s|p) - \int_v^{b'} (s-v)[1+\lambda(1-H(b'|s))]\mathrm{d}H(s|p) - \int_{b'}^{b} (s-v)(1+\lambda)\mathrm{d}H(s|p). \end{cases}$$

Then, $b' = v$ is a PE because $U(A(p, b')|A(p, b')) \geq U(A(p, b)|A(p, b'))$ for all b and p ((1b) and (2b)), $b' < v$ is not a PE because $U(A(p, b')|A(p, b')) < U(A(p, b)|A(p, b'))$ for $b = v$ for all $p \leq b'$ ((1a) and (2b)), and $b' > v$ is not a PE because $U(A(p, b')|A(p, b')) < U(A(p, b)|A(p, b'))$ for $b = v$ for all $p \leq b$ ((1c) and (1b)). Thus, $b' = v$ is the only PE and thus the unique PPE in the EA, and $\beta^*(v) = v$ constitutes the unique PE profile and the unique PPE profile. ■

Appendix D. Proofs for the Dutch Auction

Lemma 4. *If the $n-1$ other bidders bid according to $\beta(v) = a\hat{\beta}(v)$ with $a > 0$, then $k(p) = p + \frac{G(p)}{g(p)}$ is increasing and continuous for the considered bidder.*

Proof: Note that $\beta(\cdot)$ is a continuous and strictly monotonic increasing and, thus, invertible function, because the cdf F has full support on $[0, \bar{v})$.

Since $\beta(x) = a\hat{\beta}(x) = a\frac{\int_0^x s\bar{f}(s)\,\mathrm{d}s}{\bar{F}(x)}$ is continuous and strictly monotonic increasing and $\beta(0) = 0$, we can represent every p by an x via $p = \beta(x)$. It holds that $p' > p$ if and only if $x' > x$, and that

$$\beta'(x) = \frac{ax\bar{f}(x)\bar{F}(x) - a\int_0^x s\bar{f}(s)\,\mathrm{d}s\bar{f}(x)}{\bar{F}(x)^2} = \frac{(ax - \beta(x))\bar{f}(x)}{\bar{F}(x)}. \tag{D.1}$$

If all others use $\beta(v)$, it is $k(p) = p + \frac{G(p)}{g(p)} = p + \frac{\bar{F}(\beta^{-1}(p))}{\bar{f}(\beta^{-1}(p))\frac{\mathrm{d}\beta^{-1}(p)}{\mathrm{d}p}}$. Using $p = \beta(x)$ we get $\tilde{k}(x) := \beta(x) + \frac{G(\beta(x))}{g(\beta(x))} = \beta(x) + \beta'(x)\frac{\bar{F}(x)}{\bar{f}(x)} = ax$ where the last equality follows from Equation (D.1). Since $\tilde{k}(x)$ is monotone in x, $k(p)$ is monotone in p. Since $\tilde{k}(x)$ is continuous, and $\bar{F}$ and β are continuous functions, $k(p)$ is continuous. ■

Proposition 4. *Given a value v and the beliefs $G(\cdot)$, b^* is a PE in the DA if and only if*

(DA1) $U(W(p, b^*)|W(p, b^*)) \geq U(W(p, p)|W(p, b^*))$ *for all* $p \geq b^*$ *and*

(DA2) $U(W(b^*, b^*)|W(b^*, b^*)) \geq U(W(b^*, b)|W(b^*, b^*))$ *for all* $b \leq b^*$.

Proof: Using the definition of a PE (Definition 1), we show that $U(W(p, b^*)|W(p, b^*)) \geq U(W(p, b)|W(p, b^*))$ for all b and $p \geq \max\{b^*, b\}$ if and only if conditions (DA1) and (DA2) are fulfilled.
if: Express $U(W(p, b)|W(p, b^*))$ for $p \geq \max\{b, b^*\}$ as

$$\begin{aligned} &U(W(p, b^*)|W(p, b^*)) \\ &= \begin{cases} U(W(p, b)|W(p, b^*)) + G(b|p)\big(U(W(b, b^*)|W(b, b^*)) - U(W(b, b)|W(b, b^*))\big) & \text{if } b \geq b^* \\ U(W(p, b)|W(p, b^*)) + G(b^*|p)\big(U(W(b^*, b^*)|W(b^*, b^*)) - U(W(b^*, b)|W(b^*, b^*))\big) & \text{if } b < b^* \end{cases} \end{aligned}$$

using (9) and (11), and (10) and (12) for $b \leq b^*$ and (13) for $b > b^*$, as well as $G(b|p)G(b^*|b) = G(b^*|p)$ and $\ln(G((b^*|b))) = \ln(G((b^*|p))) - \ln(G((b|p)))$.

It follows that $U(W(p, b^*)|W(p, b^*)) \geq U(W(p, b)|W(p, b^*))$ for all b because by (DA1) it is $U(W(b, b^*)|W(b, b^*)) \geq U(W(b, b)|W(b, b^*))$ for $b \geq b^*$ and by (DA2) it is $U(W(b^*, b^*)|W(b^*, b^*)) \geq U(W(b^*, b)|W(b^*, b^*))$ for $b < b^*$.
only if:

For $p = b \geq b^*$, $U(W(p,b^*)|W(p,b^*)) \geq U(W(p,b)|W(p,b^*))$ is equal to (DA1).

For $p = b^* > b$, $U(W(p,b^*)|W(p,b^*)) \geq U(W(p,b)|W(p,b^*))$ is equal to (DA2). ■

Lemma 1. *Given a value v and the beliefs $G(\cdot)$, b^* is a PE*

- *only if $\frac{\partial D_1(p,b,b^*)}{\partial b} \geq 0$ for $b = p = b^*$ and $\frac{\partial D_2(b^*,b,b^*)}{\partial b} \leq 0$ for $b = b^*$*
- *if $\frac{\partial D_1(p,b,b^*)}{\partial b} \geq 0$ for all p and b with $p \geq b \geq b^*$ and $\frac{\partial D_2(b^*,b,b^*)}{\partial b} \leq 0$ for all $b \leq b^*$.*

Proof: At $p = b = b^*$, $D_1(p,b,b^*) = 0$ and (DA1) holds. A bidder who considers to marginally shift his bid upwards from b^* to b must find that sticking with b^* is better, given the reference bid b^*. Thus, $D_1(\cdot)$ must not decrease by the considered shift from b^* to b. If we fix p, and $D_1(\cdot)$ is not decreased by any marginal upwards shift of the bid b for any $p \geq b \geq b^*$, and if this holds for all $p \geq b^*$, then (DA1) holds. Similarly, $D_2(b^*,b,b^*) = 0$ at $p = b = b^*$ and (DA2) holds. A marginal shift of b downwards must not increase the bidder's utility. Thus, the derivative of $D_2(\cdot)$ at $b = b^*$ must not be strictly positive. If $D_2(\cdot)$ is not decreased by any marginal downwards shift of the bid b for any $b \leq b^*$, then (DA2) holds. ■

Proposition 5. *Given a value v and the beliefs $G(\cdot)$, it holds that*

(a) b^* *is a PE* $\implies \frac{(1+\lambda_g)v-(1+\gamma_m)\frac{G(b^*)}{g(b^*)}}{1+\gamma_m} \geq b^* \geq \frac{(1+\gamma_g+\lambda_g)v-(1+\lambda_m)\frac{G(b^*)}{g(b^*)}}{1+\gamma_m+\lambda_m}$

(b) $\frac{(1+\lambda_g)v-(1+\gamma_m)\frac{G(b^*)}{g(b^*)}}{1+\gamma_m} \geq b^* \geq \frac{(1+\gamma_g+\lambda_g)v-(1+\lambda_m)\frac{G(b^*)}{g(b^*)}}{1+\lambda_m} \implies b^*$ *is a PE*

(c) *if* $\lambda_g \geq \gamma_m$, b^* *is a PE* $\iff \frac{(1+\lambda_g)v-(1+\gamma_m)\frac{G(b^*)}{g(b^*)}}{1+\gamma_m} \geq b^* \geq \frac{(1+\gamma_g+\lambda_g)v-(1+\lambda_m)\frac{G(b^*)}{g(b^*)}}{1+\gamma_m+\lambda_m}$

Proof: Part (a) follows directly from Lemma 1. The left and right inequalities have to hold if b^* is a PE because the inequalities equal conditions (17) and (16), which are necessary conditions for a PE.

To prove part (b) we will first show that, given beliefs $G(\cdot)$, (DA1) is fulfilled for all $b^* \geq \frac{(1+\gamma_g+\lambda_g)v-(1+\lambda_m)\frac{G(b^*)}{g(b^*)}}{1+\lambda_m}$. Then we will show that (DA2) is fulfilled for all $b^* \leq \frac{(1+\lambda_g)v-(1+\gamma_m)\frac{G(b^*)}{g(b^*)}}{1+\gamma_m}$.

(DA1) holds for $b^* = \hat{b}^* := \frac{(1+\gamma_g+\lambda_g)v-(1+\lambda_m)\frac{G(\hat{b}^*)}{g(\hat{b}^*)}}{1+\lambda_m}$, which has the implicit form

$$(1+\lambda_m)\left(\hat{b}^* + \frac{G(\hat{b}^*)}{g(\hat{b}^*)}\right) - (1+\gamma_g+\lambda_g)\,v = 0. \tag{D.2}$$

At $p \geq b = \hat{b}^*$, $D_1(p,\hat{b}^*,\hat{b}^*) = 0$. It follows that $D_1(p,b,\hat{b}^*) \geq 0$ for all $p \geq b > \hat{b}^*$, because $\frac{\partial D_1(p,b,\hat{b}^*)}{\partial b} \geq 0$ for all $p \geq b > \hat{b}^*$. To see this, notice that

$$\begin{aligned}\frac{\partial D_1(p,b,b^*)}{\partial b} &= -\frac{((1+\gamma_g+\lambda_g G(b^*|b))\,v - (1+\lambda_m)\,b - \gamma_m b^* G(b^*|b))\,g(b) - (1+\lambda_m)G(b)}{G(p)} \geq 0 \\ &\iff (1+\lambda_m)\left(b + \frac{G(b)}{g(b)}\right) - (1+\gamma_g+\lambda_g G(b^*|b))\,v \geq -\gamma_m b^* G(b^*|b). \end{aligned} \tag{D.3}$$

The right-hand side of (D.3) is non-positive. The left-hand side of (D.3) equals zero at $b = b^* = \hat{b}^*$ because of $G(b^*|b^*) = 1$ and (D.2); and it is non-negative for all $b > \hat{b}^*$ because $b + \frac{G(b)}{g(b)}$ increases in b by Assumption 3, and because $G(b^*|b)$ decreases in b.

(DA1) holds for $b^* > \hat{b}^*$. At $b = b^*$ the left-hand side of (D.3) equals $(1+\lambda_m)\left(b + \frac{G(b)}{g(b)}\right) - (1+\gamma_g+\lambda_g)\, v$. This term is non-negative for all $b > \hat{b}^*$ because, as we know from above, it is zero at $b = \hat{b}^*$ and because $b+\frac{G(b)}{g(b)}$ increases in b by Assumption 3. Thus, $\frac{\partial D_1(p,b,b^*)}{\partial b} \geq 0$ for all $p \geq b > b^*$. This implies $D_1(p, b, b^*) \geq 0$ for all $p \geq b > b^*$ because $D_1(p, b^*, b^*) = 0$.

(DA2) holds for $b^* = \bar{b}^* := \frac{(1+\lambda_g)v-(1+\gamma_m)\frac{G(\bar{b}^*)}{g(\bar{b}^*)}}{1+\gamma_m}$. At $p = b = \bar{b}^*$, $D_2(\bar{b}^*, \bar{b}^*, \bar{b}^*) = 0$. It follows that $D_2(\bar{b}^*, b, \bar{b}^*) \geq 0$ for all $b < \bar{b}^*$ because $\frac{\partial D_2(\bar{b}^*,b,\bar{b}^*)}{\partial b} \leq 0$ for all $b < \bar{b}^*$. To see this, notice that

$$\begin{aligned} \frac{\partial D_2(p, b, b^*)}{\partial b} &= -\frac{((1 + \lambda_g)v - (1 + \gamma_m)b)g(b) - (1 + \gamma_m)G(b)}{G(b^*)} \leq 0 \\ &\iff (1 + \lambda_g)v - (1 + \gamma_m)\left(b + \frac{G(b)}{g(b)}\right) \geq 0. \end{aligned} \tag{D.4}$$

The left-hand side of (D.4) equals zero at $b = b^* = \bar{b}^*$ and is non-negative for all $b < \bar{b}^*$ because, by Assumption 3, $b + \frac{G(b)}{g(b)}$ decreases if b decreases.

(DA2) holds for $b^* < \bar{b}^*$. We have already shown that $\frac{\partial D_2(\bar{b}^*,b,\bar{b}^*)}{\partial b} \leq 0$ for all $b < \bar{b}^*$. Thus, it is non-negative at b^* and for all $b < b^*$. This implies $D_2(b^*, b, b^*) \geq 0$ for all $b < b^*$ because $D_2(b^*, b^*, b^*) = 0$.

To prove part (c) we only have to show that (DA1) is fulfilled for all $b^* \geq \frac{(1+\gamma_g+\lambda_g)v-(1+\lambda_m)\frac{G(b^*)}{g(b^*)}}{1+\gamma_m+\lambda_m}$. (DA1) holds for $b^* = \hat{b}^* := \frac{(1+\gamma_g+\lambda_g)v-(1+\lambda_m)\frac{G(\hat{b}^*)}{g(\hat{b}^*)}}{1+\gamma_m+\lambda_m}$, which has the implicit form

$$(1 + \lambda_m)\left(\hat{b}^* + \frac{G(\hat{b}^*)}{g(\hat{b}^*)}\right) - (1 + \gamma_g + \lambda_g)\, v + \gamma_m\hat{b}^* = 0. \tag{D.5}$$

At $p \geq b = \hat{b}^*$, $D_1(p, \hat{b}^*, \hat{b}^*) = 0$. It follows that $D_1(p, b, \hat{b}^*) \geq 0$ for all $p \geq b > \hat{b}^*$, because $\frac{\partial D_1(p,b,\hat{b}^*)}{\partial b} \geq 0$ for all $p \geq b > \hat{b}^*$. According to (D.3), $\frac{\partial D_1(p,b,b^*)}{\partial b} \geq 0$ can also be expressed as

$$(1 + \lambda_m)\left(b + \frac{G(b)}{g(b)}\right) - (1 + \gamma_g + \lambda_g G(b^*|b))\, v + \gamma_m b^* G(b^*|b) \geq 0. \tag{D.6}$$

The left-hand side of (D.6) equals zero at $b = b^* = \hat{b}^*$ because of $G(b^*|b^*) = 1$ and (D.5); and it is non-negative for all $b > \hat{b}^*$ because $b + \frac{G(b)}{g(b)}$ increases in b by Assumption 3, because $G(b^*|b)$ decreases in b, and because $\gamma_m b^* \leq \lambda_g v$. The latter follows from $b^* \leq \bar{b}^* = \frac{(1+\lambda_g)v-(1+\gamma_m)\frac{G(\bar{b}^*)}{g(\bar{b}^*)}}{1+\gamma_m}$, which implies $(1 + \gamma_m)b^* \leq (1 + \lambda_g)v$ and $\gamma_m b^* \leq \lambda_g v$ for $\lambda_g \geq \gamma_m$.

To show that (DA1) also holds for $b^* > \hat{b}^*$, we argue in the same way as in the proof of (b). ■

Proposition 6. *Given beliefs $G(\cdot)$ it holds that*

(a) *if $(\lambda_m - \gamma_m)(1+\lambda_g) \geq \gamma_g(1+\gamma_m)$, then a PE exists,*

(b) *if $\lambda_m(1+\lambda_g) < \gamma_g(1+\gamma_m)$, then a PE does not exist.*

Proof: According to Proposition 5(b), a PE b^* exists if

$$\begin{aligned}
& \frac{(1+\lambda_g)v - (1+\gamma_m)\frac{G(b^*)}{g(b^*)}}{1+\gamma_m} \geq \frac{(1+\gamma_g+\lambda_g)v - (1+\lambda_m)\frac{G(b^*)}{g(b^*)}}{1+\lambda_m} \\
\iff \quad & \frac{1+\lambda_g}{1+\gamma_m} \geq \frac{1+\gamma_g+\lambda_g}{1+\lambda_m} \\
\iff \quad & (\lambda_m - \gamma_m)(1+\lambda_g) \geq \gamma_g(1+\gamma_m).
\end{aligned}$$

According to Proposition 5(a), a b that fulfills the necessary condition for a PE does not exist if

$$\begin{aligned}
& \frac{(1+\lambda_g)v - (1+\gamma_m)\frac{G(b)}{g(b)}}{1+\gamma_m} < \frac{(1+\gamma_g+\lambda_g)v - (1+\lambda_m)\frac{G(b)}{g(b)}}{1+\gamma_m+\lambda_m} \\
\Longleftarrow \quad & \frac{(1+\lambda_g)v - (1+\gamma_m)\frac{G(b)}{g(b)}}{1+\gamma_m} < \frac{(1+\gamma_g+\lambda_g)v - (1+\gamma_m+\lambda_m)\frac{G(b)}{g(b)}}{1+\gamma_m+\lambda_m} \\
\iff \quad & \frac{1+\lambda_g}{1+\gamma_m} < \frac{1+\gamma_g+\lambda_g}{1+\gamma_m+\lambda_m} \\
\iff \quad & \lambda_m(1+\lambda_g) < \gamma_g(1+\gamma_m).
\end{aligned}$$

■

Lemma 1. *Given a value v and the beliefs $G(\cdot)$, if b^* is a PE, then a unique maximum PE $\bar{b} = \frac{(1+\lambda_g)v-(1+\gamma_m)\frac{G(\bar{b})}{g(\bar{b})}}{1+\gamma_m} \geq b^*$ exists and every $b \in [b^*, \bar{b}]$ is a PE.*

Proof: Consider a PE b^*. By Proposition 5(a), $b^* \leq \frac{(1+\lambda_g)v-(1+\gamma_m)\frac{G(b^*)}{g(b^*)}}{1+\gamma_m}$, a necessary condition for b^* to be a PE. There exists a unique maximum bid $\bar{b} \geq b^*$ such that $\bar{b} = \frac{(1+\lambda_g)v-(1+\gamma_m)\frac{G(\bar{b})}{g(\bar{b})}}{1+\gamma_m}$ because $b \leq \frac{(1+\lambda_g)v-(1+\gamma_m)\frac{G(b)}{g(b)}}{1+\gamma_m} \iff b + \frac{G(b)}{g(b)} \leq \frac{1+\lambda_g}{1+\gamma_m}v$ and the left-hand side increases in b while the right-hand side is independent of b. For the same reason, $b \leq \frac{(1+\lambda_g)v-(1+\gamma_m)\frac{G(b)}{g(b)}}{1+\gamma_m}$ for any $b \in [b^*, \bar{b}]$, that is, any such b fulfills the part of the sufficient condition for b to be a PE in Proposition 5(b) that relates to the upper bound.

Because b^* is a PE, we know by Lemma 1 that $\frac{\partial D_2(b^*,b,b^*)}{\partial b} \leq 0$ at $b = b^*$, which is by (D.4) equivalent to $(1+\lambda_g)v - (1+\gamma_m)\left(b^* + \frac{G(b^*)}{g(b^*)}\right) \geq 0$. But then $(1+\lambda_g)v - (1+\gamma_m)\left(b + \frac{G(b)}{g(b)}\right) \geq 0$ for all $b < b^*$ because $b + \frac{G(b)}{g(b)}$ decreases if b decreases by Assumption 3. Therefore, the part of the sufficient condition in Lemma 1 that determines the lower bound for a b to be a PE, $\frac{\partial D_2(b,b',b)}{\partial b'} \leq 0$ for all $b' \leq b$, is fulfilled for all $b \in [b^*, \bar{b}]$. ■

Proposition 7. *If and only if* $(\lambda_m - \gamma_m)(1+\lambda_g) \geq \gamma_g(1+\gamma_m)$ *there exist symmetric PE profiles with* $\beta(v) \in [\underline{\beta}(v), \bar{\beta}(v)]$ *and with the monotone interval boundaries*

$$\underline{\beta}(v) = \frac{1+\gamma_g+\lambda_g}{1+\lambda_m}\,\hat{\beta}(v) \quad \textit{and} \quad \bar{\beta}(v) = \frac{1+\lambda_g}{1+\gamma_m}\,\hat{\beta}(v).$$

Moreover, given v*, for any* $\beta(v) \in [\underline{\beta}(v), \bar{\beta}(v)]$ *there exists a symmetric PE profile in which* $\beta(v)$ *is chosen.*

Proof: The condition $(\lambda_m - \gamma_m)(1+\lambda_g) \geq \gamma_g(1+\gamma_m)$ is necessary for the existence of such an interval of PE profiles because if it does not hold then $\underline{\beta}(v) > \bar{\beta}(v)$.

Next we prove sufficiency. Note, $(\lambda_m - \gamma_m)(1+\lambda_g) \geq \gamma_g(1+\gamma_m)$ if and only if $\underline{\beta}(v) \leq \bar{\beta}(v)$.

We first show that a symmetric PE profile with $\underline{\beta}(v)$ fulfills (DA1), that is, the part of the sufficient condition for existence of a PE that relates to the lower bound in Proposition 5(b), for all bidders. A bidding function $\beta(v)$ fulfills (DA1) if for all v

$$\beta(v) \geq \frac{(1+\lambda_g+\gamma_g)v - (1+\lambda_m)\frac{G(\beta(v))}{g(\beta(v))}}{1+\lambda_m}. \tag{D.7}$$

As argued in the proof of Proposition 5(b), monotonicity of $k(p) = p + \frac{G(p)}{g(p)}$ (Lemma 4) assures that the smallest $\beta(v)$ for which (D.7) holds for all v is $\underline{\beta}(v) = \frac{(1+\lambda_g+\gamma_g)v-(1+\lambda_m)\frac{G(\underline{\beta}(v))}{g(\underline{\beta}(v))}}{1+\lambda_m}$. Rearranging, we get

$$\big((1+\lambda_g+\gamma_g)v - (1+\lambda_m)\underline{\beta}(v)\big)\, g(\underline{\beta}(v)) - (1+\lambda_m)G(\underline{\beta}(v)) = 0.$$

Using symmetry, that is, all bidders choosing $\underline{\beta}(v)$, and monotonicity of $\underline{\beta}(v)$, we can, for any v and $b = \underline{\beta}(v)$, replace $G(\underline{\beta}(v)) = G(b) = \bar{F}(\underline{\beta}^{-1}(b)) = \bar{F}(v)$ and $g(\underline{\beta}(v)) = g(b) = \frac{\mathrm{d}G(b)}{\mathrm{d}b} = \frac{\mathrm{d}\bar{F}(\underline{\beta}^{-1}(b))}{\mathrm{d}b} = \frac{\bar{f}(v)}{\underline{\beta}'(v)}$ where $\underline{\beta}'(v) = \frac{\mathrm{d}\underline{\beta}(v)}{\mathrm{d}v}$ to get

$$\begin{aligned}
&\big((1+\lambda_g+\gamma_g)v - (1+\lambda_m)\underline{\beta}(v)\big)\frac{\bar{f}(v)}{\underline{\beta}'(v)} - (1+\lambda_m)\bar{F}(v) = 0 \\
\iff\ &(1+\lambda_m)\left(\bar{f}(v)\underline{\beta}(v) + \bar{F}(v)\underline{\beta}'(v)\right) - (1+\lambda_g+\gamma_g)v\bar{f}(v) = 0.
\end{aligned} \tag{D.8}$$

Solving for the unique solution of the differential equation (D.8) for $\underline{\beta}(0) = 0$, using

$$\begin{aligned}
&\frac{\mathrm{d}(1+\lambda_m)\bar{F}(v)\underline{\beta}(v)}{\mathrm{d}v} = (1+\lambda_g+\gamma_g)v\bar{f}(v) \\
\implies\ &(1+\lambda_m)\bar{F}(v)\underline{\beta}(v) = (1+\lambda_g+\gamma_g)\int_0^v s\bar{f}(s)\,\mathrm{d}s,
\end{aligned}$$

gives

$$\underline{\beta}(v) = \frac{1+\lambda_g+\gamma_g}{1+\lambda_m} \frac{\int_0^v s\bar{f}(s)\,\mathrm{d}s}{\bar{F}(v)} = \frac{1+\gamma_g+\lambda_g}{1+\lambda_m} \hat{\beta}(v).$$

Second, along the same lines we show that a symmetric PE profile with $\bar{\beta}(v)$ fulfills (DA2), that is, the part of the sufficient conditions for existence of a PE in Proposition 5(b) that relates to the upper bound, for all bidders. A bidding function $\beta(v)$ fulfills (DA2) if (and only if) for all v

$$\beta(v) \leq \frac{(1+\lambda_g)v - (1+\gamma_m)\frac{G(\beta(v))}{g(\beta(v))}}{1+\gamma_m}. \tag{D.9}$$

As argued in the proof of Proposition 5(b), monotonicity of $k(p) = p + \frac{G(p)}{g(p)}$ (Lemma 4) assures that the largest $\beta(v)$ for which (D.9) holds for all v is $\bar{\beta}(v) = \frac{(1+\lambda_g)v-(1+\gamma_m)\frac{G(\bar{\beta}(v))}{g(\bar{\beta}(v))}}{1+\gamma_m}$. Rearranging, we get

$$\big((1+\lambda_g)v - (1+\gamma_m)\bar{\beta}(v)\big)\, g(\bar{\beta}(v)) - (1+\gamma_m)G(\bar{\beta}(v)) = 0.$$

Using symmetry and monotonicity, and replacing $G(\bar{\beta}(v)) = \bar{F}(v)$ and $g(\bar{\beta}(v)) = \frac{\bar{f}(v)}{\bar{\beta}'(v)}$ as above, we get

$$\begin{aligned} &\big((1+\lambda_g)v - (1+\gamma_m)\bar{\beta}(v)\big)\, \frac{\bar{f}(v)}{\bar{\beta}'(v)} - (1+\gamma_m)\bar{F}(v)) = 0 \\ \iff\; &(1+\gamma_m)\left(\bar{f}(v)\bar{\beta}(v) + \bar{F}(v)\bar{\beta}'(v)\right) - (1+\lambda_g)v\bar{f}(v) = 0. \end{aligned} \tag{D.10}$$

Solving for the unique solution of the differential equation (D.10) for $\bar{\beta}(0) = 0$, using

$$\begin{aligned} \frac{\mathrm{d}(1+\gamma_m)\bar{F}(v)\bar{\beta}(v)}{\mathrm{d}v} &= (1+\lambda_g)v\bar{f}(v) \\ \implies (1+\gamma_m)\bar{F}(v)\bar{\beta}(v) &= (1+\lambda_g)\int_0^v s\bar{f}(s)\,\mathrm{d}s, \end{aligned}$$

gives

$$\bar{\beta}(v) = \frac{1+\lambda_g}{1+\gamma_m} \frac{\int_0^v s\bar{f}(s)\,\mathrm{d}s}{\bar{F}(v)} = \frac{1+\lambda_g}{1+\gamma_m} \hat{\beta}(v).$$

Combining the two steps, we find that both $\underline{\beta}(v)$ and $\bar{\beta}(v)$ fulfill (DA1) and (DA2) and constitute a symmetric PE if adopted by all bidders, because $\underline{\beta}(v) \leq \bar{\beta}(v)$ for all v and because $\underline{\beta}(v)$ is the smallest bidding function that, if adopted by all bidders, fulfills (DA1), and $\bar{\beta}(v)$ is the largest bidding function that, if adopted by all bidders, fulfills (DA2).

It remains to show that $[\underline{\beta}(v), \bar{\beta}(v)]$ is an interval of PE that constitute symmetric PE profiles. We will prove that each member of the family of monotone bidding functions $\beta(v) = a\hat{\beta}(v)$ with $a \in [\frac{1+\lambda_g+\gamma_g}{1+\lambda_m}, \frac{1+\lambda_g}{1+\gamma_m}]$ constitutes a symmetric PE profile if adopted by all bidders, that is, it fulfills (DA1) and

(DA2) with beliefs dictated by symmetry of bidding functions and distributions of values.

Consider a symmetric PE profile with $\beta(v) = a\hat{\beta}(v)$. First, for any $a \geq \frac{1+\lambda_g+\gamma_g}{1+\lambda_m}$, this profile fulfills (DA1): For all $p = \beta(x) \geq \beta(v)$ is $D_1(\beta(x), \beta(x), \beta(v)) = U(W(\beta(x), \beta(v))|W(\beta(x), \beta(v))) - U(W(\beta(x), \beta(x))|W(\beta(x), \beta(v))) \geq 0$ because at $\beta(x) = \beta(v)$ it is $D_1(\beta(x), \beta(x), \beta(v)) = 0$ and for all $p \geq \beta(x) \geq \beta(v)$ it is $\frac{\partial D_1(p,\beta(x),\beta(v))}{\partial x} = -\frac{\partial U(W(p,\beta(x))|W(p,\beta(v)))}{\partial x} \geq 0$ for any $a \geq \frac{1+\lambda_g+\gamma_g}{1+\lambda_m}$. This holds because for $x \geq v$

$$
\begin{aligned}
&\frac{\partial U(W(p, \beta(x))|W(p, \beta(v)))}{\partial x} \\
&= \frac{\partial \frac{1}{G(p)} \left((v - \beta(x))F(x) + \lambda_g v F(v) \ln \frac{F(x)}{G(p)} + \gamma_g v(F(x) - F(v)) - \lambda_m(\beta(x)F(x) - \beta(v)F(v))\right)}{\partial x} \\
&= \frac{1}{G(p)} \left((v - \beta(x))f(x) - \beta'(x)F(x) + \lambda_g v F(v)\frac{f(x)}{F(x)} + \gamma_g v f(x) - \lambda_m(\beta'(x)F(x) + \beta(x)f(x))\right) \\
&= \frac{1}{G(p)} \left(\left(1 + \lambda_g \frac{F(v)}{F(x)} + \gamma_g\right) v f(x) - (1 + \lambda_m) \left(\beta'(x)F(x) + \beta(x)f(x)\right)\right) \\
&= \frac{f(x)}{G(p)} \left(\left(1 + \lambda_g \frac{F(v)}{F(x)} + \gamma_g\right) v - (1 + \lambda_m)ax\right) \\
&= \frac{f(x)}{G(p)} \left((1 + \lambda_g + \gamma_g)(v - x) + \lambda_g \left(\frac{F(v)}{F(x)} - 1\right) v + (1 + \lambda_g + \gamma_g - (1 + \lambda_m)a)\, x\right) \leq 0,
\end{aligned}
$$

where the forth equality uses $\beta'(x) = \frac{(ax-\beta(x))f(x)}{F(x)}$ (see (D.1)). The inequality in the last line holds because the all three terms in the large parentheses are non-positive.

Second, for any $a \geq \frac{1+\lambda_g+\gamma_g}{1+\lambda_m}$, this profile fulfills (DA2): For all $\beta(x) \leq \beta(v)$ is $D_2(\beta(v), \beta(x), \beta(v)) = U(W(\beta(v), \beta(v))|W(\beta(v), \beta(v))) - U(W(\beta(v), \beta(x))|W(\beta(v), \beta(v))) \geq 0$ because at $\beta(x) = \beta(v)$ it is $D_2(\beta(v), \beta(x), \beta(v)) = 0$ and for all $\beta(x) \leq \beta(v)$ it is $\frac{\partial D_2(\beta(v),\beta(x),\beta(v))}{\partial x} = -\frac{\partial U(W(\beta(v),\beta(x))|W(\beta(v),\beta(v)))}{\partial x} \leq 0$ for any $a \leq \frac{1+\lambda_g}{1+\gamma_m}$. This holds because for $x \leq v$

$$
\begin{aligned}
&\frac{\partial U(W(\beta(v), \beta(x))|W(\beta(v), \beta(v)))}{\partial x} \\
&= \frac{\partial \left((v - \beta(x))\frac{F(x)}{F(v)} - \lambda_g v \left(1 - \frac{F(x)}{F(v)}\right) + \gamma_m \left(\beta(v) - \beta(x)\frac{F(x)}{F(v)}\right)\right)}{\partial x} \\
&= \frac{1}{F(v)} \left((v - \beta(x))f(x) - \beta'(x)F(x) + \lambda_g v f(x) - \gamma_m(\beta'(x)F(x) + \beta(x)f(x))\right) \\
&= \frac{1}{F(v)} \left((1 + \lambda_g)v f(x) - (1 + \gamma_m)(\beta'(x)F(x) + \beta(x)f(x))\right) \\
&= \frac{1}{F(v)} \left(((1 + \lambda_g)v - (1 + \gamma_m)ax)\, f(x)\right) \\
&= \frac{1}{F(v)} \big(\left((1 + \lambda_g)(v - x) + (1 + \lambda_g - (1 + \gamma_m)a)\, x\right) f(x)\big) \geq 0,
\end{aligned}
$$

where the forth equality uses $\beta'(x) = \frac{(ax-\beta(x))f(x)}{F(x)}$. The inequality in the last line holds because both terms in the large parentheses are non-negative. ■

Proposition 8. *In the DA, if a symmetric PE profile* $(\beta(v), \dots, \beta(v))$ *exists, then* $\beta(v) \leq \bar{\beta}(v)$.

Proof: All bidders choosing $\bar{\beta}(v)$ constitutes the largest possible symmetric PE profile because (D.9) in the proof of Proposition 7 is necessary and sufficient for (DA2) to be met and because in the same proof we find that any $\beta(v)$ that constitutes a symmetric PE and for which (D.9) holds must be weakly smaller than $\bar{\beta}(v)$. ■

Lemma 2. *In the DA, the smallest symmetric PE profile that can exist is constituted by the monotonic bidding function*

$$\beta_{min}(v) = a\frac{\int_0^v s\bar{f}(s)\bar{F}(s)^{c-1}\, ds}{\bar{F}(v)^c} = \frac{a}{c}\left(v - \frac{\int_0^v \bar{F}(s)^c\, ds}{\bar{F}(v)^c}\right)$$
$$\textit{with } a = \frac{1+\lambda_g+\gamma_g}{1+\lambda_m} \textit{ and } c = \frac{1+\lambda_m+\gamma_m}{1+\lambda_m}$$

Proof: The bidding function $\beta_{min}(v)$ is monotone because

$$\begin{aligned}\frac{\mathrm{d}\beta_{min}(v)}{\mathrm{d}v} &= \frac{av\bar{f}(v)\bar{F}^{2c-1} - ab\bar{F}^{c-1}(v)\bar{f}(v)\int_0^v s\bar{f}(s)\bar{F}(s)^{c-1}\,\mathrm{d}s}{\bar{F}(v)^{2c}} \\ &= \frac{(av - c\beta_{min}(v))\bar{f}(v)}{\bar{F}(v)} = \frac{a\bar{f}(v)\int_0^v \bar{F}(s)^{c-1}\,\mathrm{d}s}{\bar{F}(v)^{c+1}} > 0.\end{aligned}$$

According to Proposition 5, for a given value v and the beliefs $G(\cdot)$, the smallest possible PE, described by the bidding function $\beta_{min}(v)$, is

$$\beta_{min}(v) = \frac{(1+\gamma_g+\lambda_g)v - (1+\lambda_m)\frac{G(\beta_{min}(v))}{g(\beta_{min}(v))}}{1+\gamma_m+\lambda_m}. \tag{D.11}$$

Rearranging (D.11) and using symmetry (i.e., all bidders choosing $\beta_{min}(v)$) and monotonicity of $\beta_{min}(v)$, we get

$$(1+\gamma_m+\lambda_m)\bar{f}(v)\beta_{min}(v) + (1+\lambda_m)\bar{F}(v)\frac{\mathrm{d}\beta_{min}(v)}{\mathrm{d}v} - (1+\lambda_g+\gamma_g)v\bar{f}(v) = 0.$$

Solving this differential equation for $\beta_{min}(0) = 0$ gives the unique solution

$$\beta_{min}(v) = a\frac{\int_0^v s\bar{f}(s)\bar{F}(s)^{c-1}\,\mathrm{d}s}{\bar{F}(v)^c} \text{ with } a = \frac{1+\lambda_g+\gamma_g}{1+\lambda_m} \text{ and } c = \frac{1+\lambda_m+\gamma_m}{1+\lambda_m}.$$

■

Appendix E. Proof on Comparisons Between the Auctions

Proposition 9. *Let $\gamma_m(1+\gamma_g) \leq \lambda_g(1+\lambda_m)$ and $(\lambda_m - \gamma_m)(1+\lambda_g) \geq \gamma_g(1+\gamma_m)$, i.e., symmetric PE profiles in the EA and DA exist. Compare expected revenues in symmetric PE profiles.*

(a) *The expected revenue in the EA is lower than the expected revenue in the DA:*

$$\max E[R^{EA}] \leq \min E[R^{DA}].$$

(b) *If*

- $\lambda_m > \gamma_m = 0$, *then the highest expected revenue in the EA is equal to the lowest expected revenue in the DA:*

$$\max E[R^{EA}] = \frac{1+\lambda_g+\gamma_g}{1+\lambda_m} E[V_{(2:n)}] = \min E[R^{DA}].$$

- $\lambda_m = \gamma_m = \gamma_g = 0 \leq \lambda_g$, *then the expected revenue from the EA is at least as high as with gain-loss neutral bidders but at most as high as that from the unique symmetric PE profile in the DA*

$$\min E[R^{EA}] = E[V_{(2:n)}] \leq \max E[R^{EA}] = (1+\lambda_g)E[V_{(2:n)}] = E[R^{DA}]$$

and the revenue in the PPE of the EA is lower than that in the PPE of the DA: $E[V_{(2:n)}] \leq (1+\lambda_g)E[V_{(2:n)}]$.

Proof: The condition $\gamma_m(1+\gamma_g) \leq \lambda_g(1+\lambda_m)$ is necessary and sufficient for the existence of symmetric PE profiles in the EA, $(\lambda_m - \gamma_m)(1+\lambda_g) \geq \gamma_g(1+\gamma_m)$ is sufficient for the existence of symmetric PE profiles in the DA (see Corollary 1 and Proposition 7), and $\lambda_m(1+\lambda_g) < \gamma_g(1+\gamma_m)$ is sufficient for the non-existence of PE in the DA (see Proposition 6).

The expected revenue from the symmetric Bayes equilibria with bidding functions $\hat{\beta}^{EA}(v) = v$ in the EA and $\hat{\beta}^{DA}(v) = \frac{\int_0^v sf(s)\,\mathrm{d}s}{F(v)}$ in the DA is the same and equal to $E[V_{(2:n)}]$.

Proof of (a): The sufficient condition for existence of a symmetric PE profile in the DA $(\lambda_m - \gamma_m)(1+\lambda_g) \geq \gamma_g(1+\gamma_m)$ implies $\lambda_m(1+\lambda_g) \geq \gamma_g(1+\gamma_m)$. Then, in the EA, the interval $\left[\frac{1+\gamma_g}{1+\lambda_m}v, \frac{1+\lambda_g+\gamma_g}{1+\lambda_m+\gamma_m}v\right]$ covers all $\beta^{EA}(v)$ that constitute a symmetric PE (see Proposition 2 and Equation (C.5)).

The expected revenue in the EA from the maximum symmetric PE profile with $\beta^{EA}_{max}(v) = k\hat{\beta}^{EA}(v) = kv$ with $k = \frac{1+\lambda_g+\gamma_g}{1+\lambda_m+\gamma_m}$ is $\max \mathrm{E}[R^{EA}] = k\mathrm{E}[V_{(2:n)}]$. According to Lemma 3, the smallest symmetric PE profile that can exist in the DA is constituted by

$$\beta_{min}^{DA}(v) = k\left(v - \frac{\int_0^v \bar{F}(s)^c \mathrm{d}s}{\bar{F}(v)^c}\right) \text{ with } c = \frac{1+\lambda_m+\gamma_m}{1+\lambda_m} \geq 1$$

and, thus, generates the expected revenue $\min \mathrm{E}[R^{DA}]$. Consider the bidding function

$$\tilde{\beta}^{DA}(v) = k\hat{\beta}^{DA}(v) = k\left(v - \frac{\int_0^v \bar{F}(s)\mathrm{d}s}{\bar{F}(v)}\right).$$

If all bidders choose $\tilde{\beta}^{DA}$, the expected revenue yields $\mathrm{E}[\tilde{R}^{DA}] = k\mathrm{E}[V_{(2:n)}] = \max \mathrm{E}[R^{EA}]$. It is $\beta_{min}^{DA}(v) \geq \tilde{\beta}^{DA}(v)$ for $v \geq 0$ because

$$\begin{aligned}\beta_{min}^{DA}(v) \geq \tilde{\beta}^{DA}(v) \iff & \frac{\int_0^v \bar{F}(s)^c \mathrm{d}s}{\bar{F}(v)^c} \leq \frac{\int_0^v \bar{F}(s)\mathrm{d}s}{\bar{F}(v)} \\ \Longleftarrow \frac{\bar{F}(s)^c}{\bar{F}(v)^c} \leq \frac{\bar{F}(s)}{\bar{F}(v)}\ \forall s \in [0,v] \iff & \bar{F}(s) \leq \bar{F}(v)\ \forall s \in [0,v].\end{aligned}$$

Therefore, $\min \mathrm{E}[R^{DA}] \geq \mathrm{E}[\tilde{R}^{DA}] = \max \mathrm{E}[R^{EA}]$.

Proof of (b): If $\gamma_m = 0$, the sufficient condition for existence of a symmetric PE profile in the DA $(\lambda_m - \gamma_m)(1+\lambda_g) \geq \gamma_g(1+\gamma_m)$ equals the necessary condition $\lambda_m(1+\lambda_g) \geq \gamma_g(1+\gamma_m)$. Then, the interval $\left[\frac{1+\gamma_g+\lambda_g}{1+\lambda_m}\,\hat{\beta}(v), (1+\lambda_g)\,\hat{\beta}(v)\right]$ covers all $\beta^{DA}(v)$ that constitute a symmetric PE profile by Proposition 7. The minimum expected revenue corresponds to the lowest bids and equals $\frac{1+\lambda_g+\gamma_g}{1+\lambda_m} E[V_{(2:n)}]$.

If $\lambda_m(1+\lambda_g) \geq \gamma_g(1+\gamma_m)$, then, in the EA, the interval $\left[\frac{1+\gamma_g}{1+\lambda_m}v, \frac{1+\lambda_g+\gamma_g}{1+\lambda_m+\gamma_m}v\right]$ covers all $\beta^{EA}(v)$ that constitute a symmetric PE (see Proposition 2 and Equation (C.5)). If $\gamma_m = 0$, the highest expected revenue in the EA is therefore $\frac{1+\lambda_g+\gamma_g}{1+\lambda_m} E[V_{(2:n)}]$, which equals the minimum expected revenue in the DA.

If additionally to $\gamma_m = 0$ also $\lambda_m = 0$, then symmetric PE profiles in the DA exist only if $\gamma_g = 0$. The interval that covers all $\beta^{DA}(v)$ that constitute a symmetric PE profile in the DA reduces to $(1+\lambda_g)\,\hat{\beta}(v)$ with the expected revenue $(1+\lambda_g)E[V_{(2:n)}]$. The interval that covers all $\beta^{EA}(v)$ that constitute a symmetric PE profile in the EA becomes $\left[\frac{1}{1+\lambda_m}v, (1+\lambda_g)\,v\right]$ with expected revenues in $\left[E[V_{(2:n)}, (1+\lambda_g)E[V_{(2:n)}]\right]$. The PPE profile in the EA provides the revenue $E[V_{(2:n)}]$ which is lower than $(1+\lambda_g)E[V_{(2:n)}]$, the revenue in the PPE of the DA.

■